

Ans

$$\hat{S}_+ = \sqrt{2S} \left(1 - \frac{\hat{a}^\dagger \hat{a}}{2S}\right)^{1/2} \hat{a}$$

$$\hat{S}_- = \sqrt{2S} \hat{a}^\dagger \left(1 - \frac{\hat{a}^\dagger \hat{a}}{2S}\right)$$

$$[\hat{S}_+, \hat{S}_-] = 2S \left[ \left(1 - \frac{\hat{a}^\dagger \hat{a}}{2S}\right)^{1/2} \hat{a}, \hat{a}^\dagger \left(1 - \frac{\hat{a}^\dagger \hat{a}}{2S}\right)^{1/2} \right]$$

$$\text{Now } \left[ \left(1 - \frac{\hat{a}^\dagger \hat{a}}{2S}\right)^{1/2} \hat{a}, \hat{a}^\dagger \left(1 - \frac{\hat{a}^\dagger \hat{a}}{2S}\right)^{1/2} \right]$$

$$= \left(1 - \frac{\hat{a}^\dagger \hat{a}}{2S}\right)^{1/2} \underbrace{\hat{a} \hat{a}^\dagger}_{\hat{a}^\dagger \hat{a} + 1} \left(1 - \frac{\hat{a}^\dagger \hat{a}}{2S}\right)^{1/2}$$

$$- \hat{a}^\dagger \left(1 - \frac{\hat{a}^\dagger \hat{a}}{2S}\right)^{1/2} \left(1 - \frac{\hat{a}^\dagger \hat{a}}{2S}\right)^{1/2} \hat{a}$$

$$\text{Let } \hat{a}^\dagger \hat{a} = \hat{n} \quad (\text{number operator}) \quad \hat{n}^\dagger = (\hat{a}^\dagger \hat{a})^\dagger = \hat{a} \hat{a}^\dagger = \hat{n} + 1$$

The commutator is

$$\left(1 - \frac{\hat{n}}{2S}\right)^{1/2} \underbrace{\hat{a} \hat{a}^\dagger}_{\hat{a}^\dagger \hat{a} + 1} \left(1 - \frac{\hat{n}}{2S}\right)^{1/2} - \hat{a}^\dagger \left(1 - \frac{\hat{n}}{2S}\right)^{1/2} \left(1 - \frac{\hat{n}}{2S}\right)^{1/2} \hat{a}$$

$$\hat{S}_+ |S_z\rangle = \sqrt{2S} \sqrt{1 - \frac{\hat{a}^\dagger \hat{a}}{2S}} \hat{a} |S_z\rangle$$

$$\hat{S}_z = -\hat{a}^\dagger \hat{a}$$

$$\hat{a} |S_z\rangle = |S_z - 1\rangle$$

$$\hat{S}_+ |n\rangle = \sqrt{2S} \sqrt{1 - \frac{\hat{a}^\dagger \hat{a}}{2S}} \hat{a} |n\rangle$$

$$= \sqrt{2S} \sqrt{n+1} \sqrt{1 - \frac{\hat{a}^\dagger \hat{a}}{2S}} |n+1\rangle$$

$$= \sqrt{2S} \sqrt{n+1} \sqrt{1 - \frac{(n+1)}{2S}} |n+1\rangle$$

$$S_z |n\rangle = (S - n) |n\rangle$$

$$S_z |n+2\rangle = (S - n + 2) |n+2\rangle$$

$$\hat{S}_z \hat{S}_+ |n\rangle = \sqrt{2S} \sqrt{n+1} \sqrt{1 - \frac{n+1}{2S}} (S - n + 1) |n+1\rangle$$

$$\hat{S}_+ \hat{S}_z |n\rangle = \hat{S}_+ (S - n) |n\rangle$$

$$= (S - n) \hat{S}_+ |n\rangle$$

$$= (S - n) \sqrt{2S} \sqrt{n+1} \sqrt{1 - \frac{(n+1)}{2S}} |n+1\rangle$$

$$[\hat{S}_z, \hat{S}_+] = \sqrt{2S} \sqrt{n+1} \sqrt{1 - \frac{n+1}{2S}} (S - n + 1 - S + n) = \sqrt{(n+1)(2S - n - 1)}$$

