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Many oscillations of a rigid rod

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The many oscillations of a rigid rod—pendular, bifilar, torsional, coupled, nonlinear (including pure x^3), and chaotic—provide interesting opportunities for experimental and theoretical investigations in introductory, intermediate, and advanced mechanics courses. Because of the rod's simple geometry, precise expressions for the periods of its oscillations can be calculated, allowing comparisons between theory and experiment to within 0.5%. At this level of precision, the finite diameter of the pin upon which the rod oscillates has an observable effect on the period. A derivation is given of this effect, and applied to a ring oscillating on rods of various diameters. The large-angle oscillations of the bifilar pendulum are shown to depart by a measurable amount from the large-angle oscillations of the physical pendulum. The unequal-arm bifilar pendulum couples torsional and pendular modes in a calculable way. The vertical bifilar pendulum is a double pendulum with calculable coupled oscillations for small amplitudes and chaotic oscillations for large amplitudes. Other oscillators that can be built from a rigid rod, including an x^3 oscillator, are described. © *1995 American Association of Physics Teachers.*

I. INTRODUCTION

In the course of developing a physical pendulum experiment for our introductory physics laboratory, I became fascinated by the many interesting oscillations that a simple circular rod can have. In addition to the more obvious bifilar and torsional oscillations, the rod can easily be caused to undergo coupled oscillations, nonlinear oscillations (including pure x^3 oscillations), and chaotic oscillations. These different oscillations are interesting in themselves, and provide opportunities for experimental and theoretical investigations in introductory, intermediate, and advanced mechanics courses. The simple geometry of the rigid rod allows for the derivation of accurate algebraic expressions for the periods of its oscillations and the generally low-energy loss in each oscillation allows for the precise measurement of these periods with a stopwatch. With a rigid rod, close comparisons of theory and experiment can be made of mechanical systems of varying degrees of complexity.

The discussion in this paper is arbitrarily limited to experiments done on a particular steel rod of length $H=22.20$ cm, radius $R=0.635$ cm, and mass $m=217$ g (Table I). The rod has three pairs of holes (radii $r=0.12$ cm) drilled along its length, the holes of each pair being equidistant from the center of the rod. The distances d_0 from the center of the rod to the centers of the holes are shown in Table I. In the equations in this paper, d is the distance from a point to the center of mass of the rod, which may be d_0, d_0+r, or d_0-r, depending on the circumstances.

The moment of inertia of the rod about its central axis is

$$I_R=mJ_R, \tag{1}$$

where $J_R=(1/2)R^2$ and the moment of inertia about the perpendicular axis through the center of mass is

$$I_H=mJ_H, \tag{2}$$

where $J_H=(1/12)H^2+(1/4)R^2$.

For the rod in Table I, the second term in Eq. (2) contributes $+0.5\%$ to J, and the six holes contribute -0.3%. These corrections are neglected here, since they are comparable to the 0.2% uncertainty in J coming from a 0.1% uncertainty in the measurement of H. For shorter or thicker rods, or holes of larger diameter, the equations in this paper should be modified by replacing $H^2/12$ with the corrected value of J.

The moments of inertia about axes parallel to the principal axes are given by the parallel axis theorem

$$J=J_{\mathrm{c.m.}}+d^2, \tag{3}$$

where $J_{\mathrm{c.m.}}$ is J_R or J_H, and d is the distance from the center of mass of the rod to the parallel axis.

For a simple pendulum, corrections to the zero-angle period from finite amplitude, mass of the supporting string, buoyancy of the bob, and effects of air flow around the bob have been shown[1] to be individually and collectively less than 0.02% and so can be neglected at the level of precision we are considering here.

II. THE PHYSICAL PENDULUM

A. Small amplitude oscillations

The rod is hung vertically on a pin through one of its holes [Fig. 1(a)]. Excellent results are obtained using a section of a small paper clip tightly clamped to a stand. With this simple arrangement, 50 or more small-angle oscillations can be measured at a time. From repeated measurements, the uncertainty in the measured period was found to be less than 0.002 s.

As the rod oscillates, the top of the hole rolls without slipping along the top of the pin, so that the point of contact of the rod with the pin is stationary. The angular frequency for small-angle oscillation about this point is[2]

$$\omega_0=\sqrt{\frac{gd}{J}}, \tag{4}$$

where $d=d_0+r$ is the distance from the center of mass of the rod to the top of the hole and mJ is the moment of inertia

Table I. Dimensions of the steel rod used in this paper. The rod has three pairs of holes, at distances d_0 from their centers to the center of the rod.

Length, H	22.20 cm	Numbered from end	Distance d_0 from center of hole to center of rod
Radius, R	0.635 cm		
Mass, m	217 g	Hole 1	10.63cm
Hole radii, r	0.12 cm	Hole 2	6.27cm
		Hole 3	1.80cm

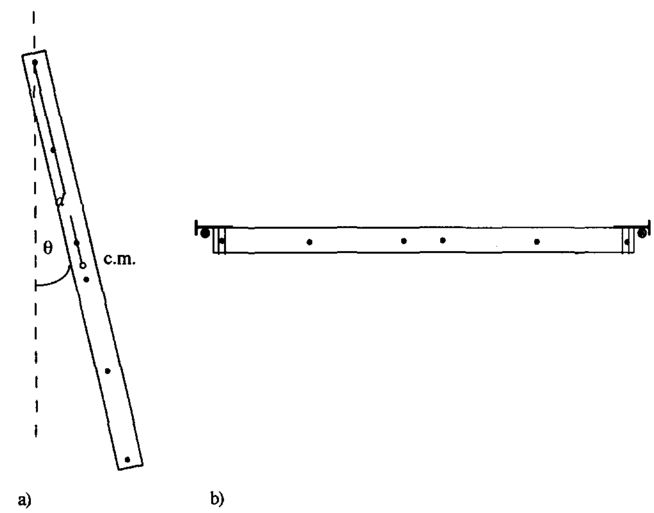

Fig. 1. The physical pendulum. (a) Rotating about an axis perpendicular to its length. (b) Rotating about an axis parallel to its length.

of the rod about an axis through the top of the hole.

From Eqs. (2)–(4), the small-amplitude period of the rod is

$$T_0=\frac{2\pi}{\omega_0}=2\pi\sqrt{\frac{(1/12)H^2+d^2}{gd}}. \tag{5}$$

Note that the period has the minimum $2\pi(H/\sqrt{3}g)^{1/2}=0.718$ s at the radius of gyration $d_g = H/\sqrt{12} = 6.41$ cm.

Table II compares the measured periods to the periods calculated from Eq. (5) with $g=980.3\ \mathrm{m/s^2}$, the acceleration of gravity in Boston, MA. These agree in the first two cases within the expected error of 0.002 s, but disagree in the third case by ten times this error.

The source of this disagreement is the finite diameter of the pin. In the Appendix, it is shown that the angular frequency of a physical pendulum oscillating on a pin of radius ρ, in a hole of radius r, is

$$\omega_0=\sqrt{\frac{g(d+\rho')}{J}}, \tag{6}$$

where

$$\rho'=\frac{\rho}{1-\rho/r}. \tag{7}$$

Table II. Measured and calculated periods of a rod oscillating on a pin through holes at different distances d from the center of mass (c.m.).

Distance, d from c.m. to top of hole (cm)	Period, T measured (s)	Period, T calculated Eq. (5) (s)	Percent error	Period, T calculated Eq. (6) (s)	Percent error
10.75	0.7636	0.7660	0.31	0.7639	0.034
6.39	0.7162	0.7184	0.31	0.7151	−0.16
1.92	0.9467	0.9689	2.29	0.9541	0.78

The radius of a small paper clip is $\rho=0.040$ cm, and the radius of the hole is 0.12 cm, so $\rho'=0.06$ cm. This distance increases the torque on the rod, but not its moment of inertia. This decreases all the calculated periods, with the largest percentage change being for the hole with the smallest d, as shown in Table II. With this correction, the first two cases agree within 0.001 s and the third case within 0.007 s. This remaining discrepancy is within the uncertainty in the calculated period due to a 0.02 cm uncertainty in the measured distance d_0 for hole 3.

B. Large-amplitude oscillations

The exact period T of a system with the equation of motion

$$\ddot{\theta}=-\omega_0^2\sin\theta$$

is given by the energy integral[3]

$$T=\frac{4}{\sqrt{2}\omega_0}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\cos\theta-\cos\theta_0}}=\frac{2}{\omega_0}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2}-\sin^2\frac{\theta}{2}}}. \tag{8}$$

With the substitution, $\sin\theta/2=(\sin\theta_0/2)\sin x$, this becomes

$$T=\frac{4}{\omega_0}\int_0^{\pi/2}\frac{dx}{\sqrt{1-\sin^2\frac{\theta_0}{2}\sin^2 x}}=T_0\frac{2}{\pi}K\left(\sin^2\frac{\theta_0}{2}\right), \tag{9}$$

where $T_0=2\pi/\omega_0$ is the small-angle period and $K(m)$ is the complete elliptic integral of the first kind.[4]

For $m\leq 0.5$, $(2/\pi)K(m)$ is given to better than 1% by the expansion

$$\frac{2}{\pi}K(m)=\left[1+\left(\frac{1}{2}\right)^2 m+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2 m^2+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2 m^3+\cdots\right] \tag{10}$$

and for $(1-m)<0.1$, $(2/\pi)K(m)$ is given to better than 2% by

$$\frac{2}{\pi}K(m)=\frac{1}{\pi}\ln\left(\frac{16}{1-m}\right). \tag{11}$$

A rigid rod is especially useful for studying the period of oscillations with amplitudes greater than 90°. With the rod suspended from its outermost hole, it swung from 170° back to 156° in 1.74 s, which is between the periods of 1.87 and 1.44 s calculated from Eqs. (5), (9), and (11) for these angles.

C. Small-amplitude oscillations about an axis parallel to the rod

The rod can also oscillate about an axis parallel to its length by supporting it on two nails that are attached to the rod by rubber bands [Fig. 1(b)]. From Eqs. (1) and (3), the

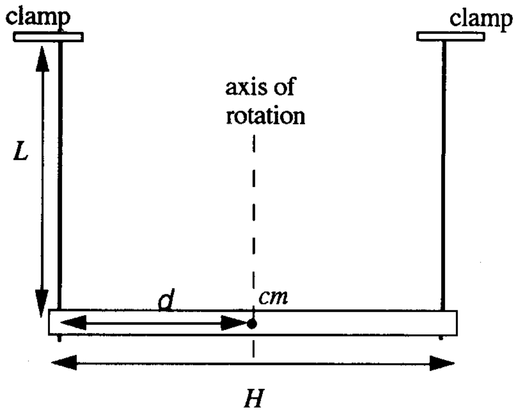

Fig. 2. The normal bifilar pendulum.

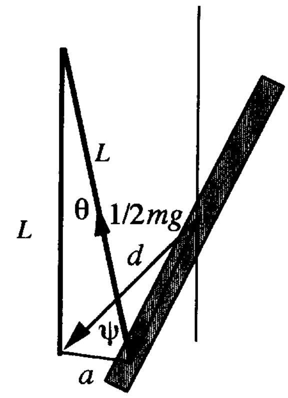

Fig. 3. Perspective view of the bifilar pendulum rotated out of the equilibrium plane of the system. The rod is horizontal and is rotated through an angle Ψ about the vertical axis through its center. For $\Psi \ll 1$, each string makes an angle $\theta=(d/L)\Psi$ with the vertical.

moment of inertia is $\frac{3}{2}mR^2$. The radius R is also the distance from the axis of rotation to the center of mass, so from Eq. (5), the period is

$$T_0=2\pi\sqrt{\frac{3R}{2g}}. \tag{12}$$

With $R=0.635$ cm, $T_0=0.20$ s, which is too small to be measured precisely by hand. Oscillations about the central axis can be measured with a bifilar (Sec. III A) or torsional (Sec. V) pendulum.

III. BIFILAR PENDULUM

In the normal bifilar pendulum, the rod is suspended horizontally from two parallel cords of equal length (Fig. 2). Interesting variations on this arrangement are discussed in Sec. III D.

A. Pendular oscillations

The rod oscillates in and out of the equilibrium plane of the system, with an angular frequency given by Eq. (4), where mJ is the moment of inertia about an axis a distance L from the central axis of the rod. Using the parallel axis theorem, this gives the period as

$$T_0=2\pi\sqrt{\frac{(1/2)R^2+L^2}{gL}}=2\pi\sqrt{\frac{L}{g}\left(1+\frac{R^2}{2L^2}\right)} \tag{13}$$

which reduces to the equation for a simple pendulum for $L \gg R$. With $L=1.605$ cm, Eq. (13) gives a period of 0.264 s, which is 4% greater than the period of a simple pendulum and in agreement with the period of 0.267 ± 0.004 s, measured with a photogate.

The rod also oscillates laterally in the equilibrium plane of the system with the period of a simple pendulum

$$T_0=2\pi\sqrt{\frac{L'}{g}}, \tag{14}$$

where $L'=L-R$ is the length of the string measured to the top of the rod. This is because the string bends at the top of the rod, allowing the rod to move parallel to itself.

Thus the two pendular modes of the bifilar have slightly different periods, which is observable for all reasonable values of L.

B. Torsional oscillations; small amplitude

The most interesting oscillation of the bifilar pendulum is the one about a vertical axis through the center of mass of the rod. For an angular twist ψ, each string makes angle θ with the vertical (Fig. 3). The tension in each string is $\frac{1}{2}mg$ and the torque each string exerts about the central axis is $\frac{1}{2}mgd\sin\theta$, where d is the distance from the point of attachment of the string to the center of mass of the rod. For small angles, the magnitude of the total torque due to both strings is

$$\tau=mgd\sin\theta\approx mgd(a/L),$$

where a is the distance the point of attachment moved. We also have $a=d\psi$, so the torque is

$$\tau=-(mgd^2/L)\psi$$

and the period of the oscillation is

$$T_0=2\pi\sqrt{\frac{I_H}{mgd^2/L}}=2\pi\sqrt{\frac{L}{g}}\frac{H/d}{\sqrt{12}}. \tag{15}$$

Since the string bends at the top of the hole—that is, the rod does not rotate about its central axis—L in Eq. (15) is measured to the top of the rod, not to the central axis.

For example, with $L=40.8$ cm and $d=6.27$ cm, the calculated period is 1.31 s, in agreement with the measured value of 1.32 s. Generalizations of Eq. (15) for nonparallel strings and strings of unequal length are given in Sec. III D.

C. Torsional oscillations; large amplitude

When the rod is rotated through an angle ψ (Fig. 3), the rod is lifted the vertical distance

$$h=L-\sqrt{L^2-a^2}, \tag{16}$$

where a is the distance the point of attachment of the string has rotated. From Fig. 3, this is

$$a=2d\sin(\psi/2).$$

With $\epsilon=2d/L$, $x=\sin(\psi/2)$, and $x_0=\sin(\psi_0/2)$, the energy integral for the period T with amplitude ψ_0 is

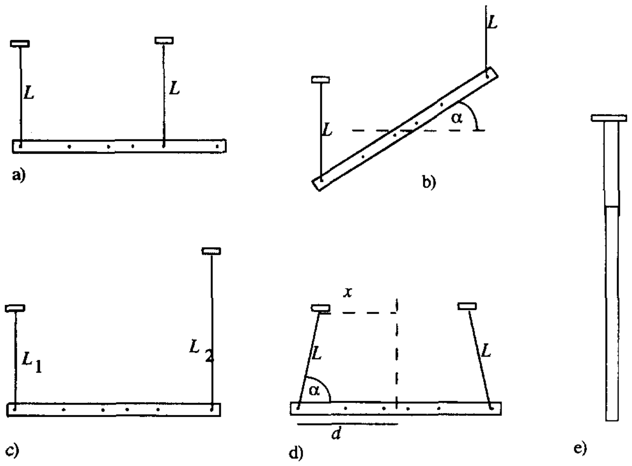

Fig. 4. Variations of the bifilar pendulum. (a) Asymmetric, (b) Inclined, (c) Unequal length, (d) Nonparallel, (e) vertical.

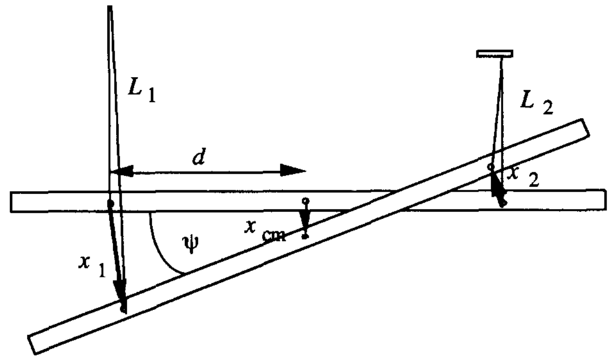

Fig. 5. The coordinates of the unequal-length bifilar pendulum. The rod is rotated through an angle Ψ about the vertical axis through its center and its center is displaced a distance $x_{\mathrm{c.m.}}$ out of the equilibrium plane of the system. The points where the string attach to the rod are displaced the distances x_1 and x_2 from the equilibrium plane.

$$T=\frac{4}{\sqrt{2}}\sqrt{\frac{I}{mgL}}\int_0^{\psi_0}[\sqrt{1-\epsilon^2x^2}-\sqrt{1-\epsilon^2x_0^2}]^{-1/2}d\psi. \tag{17}$$

From the constraints of the torsional pendulum, we have $\epsilon x<\epsilon x_0<1$. Further, by increasing L, we can make $\epsilon \ll 1$, so that $\epsilon x_0 \ll 1$ even for ψ_0 near 180°. Thus by expanding the integral in Eq. (17) in powers of $\epsilon^2x_0^2$ we get an expression that is useful for all values of ψ_0. The result, to first order in $\epsilon^2x_0^2$, is

$$T(x_0^2)/T_0=\frac{2}{\pi}K(x_0^2)\left[1-\frac{1}{8}\epsilon^2x_0^2\left(1+\frac{N(x_0^2)}{K(x_0^2)}\right)-\cdots\right], \tag{18}$$

where T_0 is the small-amplitude period given in Eq. (15), K is the complete elliptic integral of the first kind discussed in Sec. II, and N is the integral

$$N(m)=\int_0^{\pi/2}\frac{\sin^2 x}{\sqrt{1-m\sin^2 x}}\,dx$$

which has expansions similar to K. The ratio N/K equals 0.5 at $m=0$ ($\psi_0=0$), 0.53 at $m=0.5$ ($\psi_0=90°$), and 1.0 at $m=1$ ($\psi_0=180°$).

For example, with $d=1.80$ cm and $L=5.0$ cm, Eq. (18) predicts that for $\psi_0=90°$, $T/T_0=1.12$ rather than 1.18 for a simple pendulum [Eq. (9)]. Using a photogate to measure the period at 90°, the experimental value of 1.13 ± 0.02 was obtained for T/T_0.

D. Variations

Figure 4 shows five variations of the bifilar pendulum, which can be studied individually and in combinations for both small and large oscillations. Each variation requires its own analysis, which are challenges of varying difficulty.

(1) *Asymmetrical bifilar pendulum.* This is a simple variation in which the moment of inertia is about an axis parallel to the vertical axis through the center of mass [Fig. 4(a)]. This effects the period of the torsional oscillation, but not the pendular oscillations. The tension in the two cords are unequal, but this does not change the total torques. The torsional period is given by Eqs. (3) and (15).

(2) *Inclined bifilar pendulum.* In the inclined pendulum [Fig. 4(b)], the torsional oscillations are about a vertical axis that makes an angle α with the principal axis perpendicular to the rod. The moment of inertia about the vertical axis is

$$I=I_H\cos^2\alpha+I_R\sin^2\alpha. \tag{19}$$

Since $I_R \ll I_H$, for $\alpha<30°$ the period of the inclined torsional oscillator is $\cos\alpha$ times the period of the horizontal oscillator.

The inclination does not effect the pendular oscillations, provided the lengths of the two strings remain equal.

Unequal-length bifilar pendulum. This is an extremely interesting variation [Fig. 4(c)], since it couples the torsional and pendular oscillations.

For small angles, the points where the string attaches to the rod move the distances x_1 and x_2, perpendicular to the equilibrium plane of the system (Fig. 5). In terms of these variables, the position $x_{\mathrm{c.m.}}$ of the center of mass of the rod and the angular displacement ψ of the rod are

$$x_{\mathrm{c.m.}}=(x_1+x_2)/2,\quad \psi=(x_1-x_2)/2d, \tag{20}$$

where d is the distance from the center of mass to the point of attachment of each string. The kinetic and potential energies of the system are

$$K=\tfrac{1}{2}I_{\mathrm{c.m.}}\dot{\psi}^2+\tfrac{1}{2}m\dot{x}_{\mathrm{c.m.}}^2$$

and

$$U=mgy_{\mathrm{c.m.}}=\tfrac{1}{2}mg(L_1+L_2-\sqrt{L_1^2-x_1}-\sqrt{L_2^2-x_2^2}).$$

Expanding the radicals, and writing x_1 and x_2 in terms of $x_{\mathrm{c.m.}}$ and ψ, we get the small-angle Lagrangian

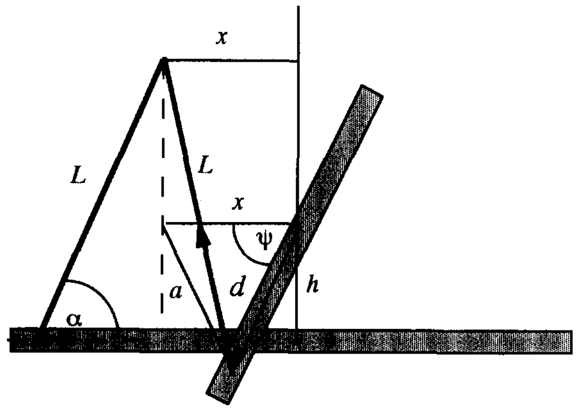

Fig. 6. Perspective view of the nonparallel bifilar pendulum, showing the rod rotated out of the equilibrium plane of the system. The rod is horizontal and is rotated through an angle ψ about the vertical axis through its center.

$$L=K-U=\frac{1}{2}\,I_{\text{c.m.}}\dot{\psi}^2+\frac{1}{2}\,m\dot{x}_{\text{c.m.}}^2-\frac{mg}{2L_{\text{eff}}}\left(x_{\text{c.m.}}^2+d^2\psi^2-\frac{\Delta L}{L_{\text{av}}}\,d\psi x_{\text{c.m.}}\right), \tag{21}$$

where

$$L_{\text{av}}=\frac{1}{2}\,(L_1+L_2),\quad L_{\text{eff}}=\frac{L_1L_2}{L_{\text{av}}},\quad \text{and } \Delta L=L_1-L_2. \tag{22}$$

The Lagrangian yields the coupled equations

$$\begin{aligned}\ddot{x}_{\text{c.m.}}&=-\omega_0^2(x_{\text{c.m.}}-\epsilon d\psi),\\ \ddot{\psi}&=-\omega_0^2\lambda\left(-\frac{\epsilon}{d}\,x_{\text{c.m.}}+\psi\right),\end{aligned} \tag{23}$$

where

$$\omega_0^2=\frac{g}{L_{\text{eff}}},\quad \lambda=\frac{md^2}{I_{\text{c.m.}}},\quad \epsilon=\frac{\Delta L}{2L_{\text{eff}}}. \tag{24}$$

Using standard methods,[5,6] the frequencies of the normal modes are found to be

$$\omega_{\pm}^2=\omega_0^2[\tfrac{1}{2}(1+\lambda)\pm\sqrt{(1-\lambda)^2/4+\lambda\epsilon^2}]. \tag{25}$$

When d is equal to the radius of gyration, λ is 1 and the equations simplify a bit. In this special case, the normal modes corresponding to ω_+ and ω_- are $x_1=0$ and $x_2=0$, respectively, which are oscillations about axes coinciding with one or the other suspending strings. For $d=6.27$ cm (hole 2), $\lambda=0.957$, and the normal modes were easily established by releasing the rod after rotating it through a small angle about a string. With $L_1=29.4$ cm and $L_2=15.7$ cm, the angular frequencies given by Eq. (25) for hole 2 are 7.91 and 5.58 s^{-1}, corresponding to periods of 0.79 and 1.13 s. These agree very well with measured periods of 0.79 and 1.09 s.

Nonparallel bifilar pendulum. Figure 4(d) shows cords of length L that are a distance $2d$ apart where they attach at the rod and a distance $2x$ apart where they attach to the supporting clamp. When the rod rotates through an angle ψ about a vertical axis (Fig. 6), its center of mass rises the height

$$h=L\cos\alpha-\sqrt{L^2-a^2}$$

where $a^2=x^2+d^2-2xd\cos\psi$. This reduces to Eq. (16) when $x=d$.

The torque is

$$\tau=-mg\,\frac{dh}{d\psi}=-\frac{xd\sin\psi}{\sqrt{L^2-(x^2+d^2-2xd\cos\psi)}}$$

so the small-amplitude period is

$$T_0==2\pi\sqrt{\frac{L}{g}\frac{H/d}{\sqrt{12}}}\;\sqrt{\frac{d\sqrt{1-\left(\frac{d-x}{L}\right)^2}}{x}}. \tag{26}$$

This reduces to Eq. (15) when $x=d$, and becomes infinite when $x=0$. That is, when the two strings meet at a point, no torque is produced by rotations about a vertical axis.

Vertical bifilar pendulum. Torsional oscillations are about the central axis of the rod [Fig. 4(e)], so I_R replaces I_H in Eq. (15). The oscillations perpendicular to the equilibrium plane have two coupled degrees of freedom; which are discussed in the next section.

IV. DOUBLE PENDULUM

A double pendulum is produced by hanging the rod vertically from a string that passes through one of its holes [Fig. 4(e)]. Besides the torsional motion described in Sec. III B, this configuration has a coupled oscillation when swinging in and out of its equilibrium plane. In one of the two normal modes, the string and rod swing together, and in the other, they swing in opposite directions. Nearly pure normal modes can be established by starting with the string and rod displaced in the same direction, or in opposite directions. Once in motion, the periods of these modes can be measured very precisely.

In this particular double pendulum, the upper arm is a massless pendulum of length L and the lower arm is a rod of length H (Fig. 7). The string attaches to the rod a distance d from the rod's center of mass. The Lagrangian of this system is[6]

$$\begin{aligned}L=&\tfrac{1}{2}mL^2\dot{\theta}_1^2+\tfrac{1}{2}(I_{\text{c.m.}}+md^2)\dot{\theta}_2^2+mdL\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2\\&+mg(L\cos\theta_1+d\cos\theta_2)\end{aligned} \tag{27}$$

which, for small oscillations, yields the coupled equations

$$\begin{aligned}\ddot{\theta}_1+\frac{d}{L}\,\ddot{\theta}_2&=-\omega_0^2\theta_1,\\ \ddot{\theta}_1+\frac{s}{L}\,\ddot{\theta}_2&=-\omega_0^2\theta_2,\end{aligned} \tag{28}$$

where

$$s=\frac{(I_{\text{c.m.}}/m)+d^2}{d}=\frac{(1/12)H^2+d^2}{d}\quad\text{and }\omega_0^2=\frac{g}{L}. \tag{29}$$

The angular frequencies of the normal modes found from these equations are

$$\omega_{\pm}^2=\frac{\omega_0^2}{2(1-d/s)}\,[(L/s+1)$$

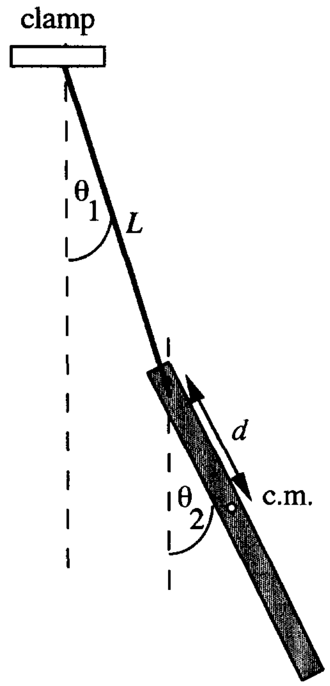

Fig. 7. Coordinates of a double pendulum.

$$\pm\sqrt{(L/s+1)^2-4(L/s)(1-d/s)}]. \tag{30}$$

For example, with $H=22.2$ cm, $L=45.4$ cm, and $d=6.27$, Eqs. (29) and (30) give 0.49 and 1.45 s for the periods of the two normal modes, compared to measured values of 0.47 and 1.47 s.

Large amplitude oscillations. It is well known by now that large-amplitude oscillations of the double pendulum can be chaotic.[7,8] To obtain chaotic motion with the rod, it is only necessary to replace the string with steel wire (Fig. 8). The

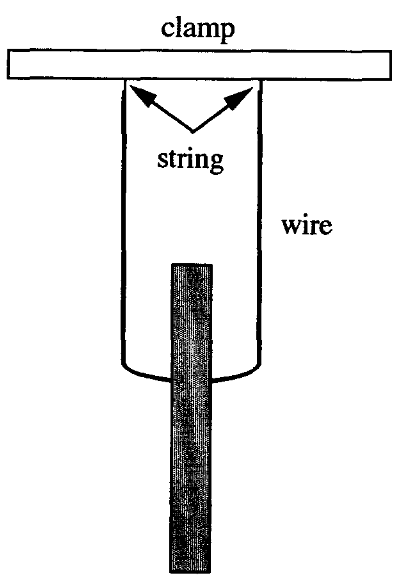

Fig. 8. The rod suspended from a steel wire. The upper arms of the wire are connected to short pieces of string that are clamped in a holder.

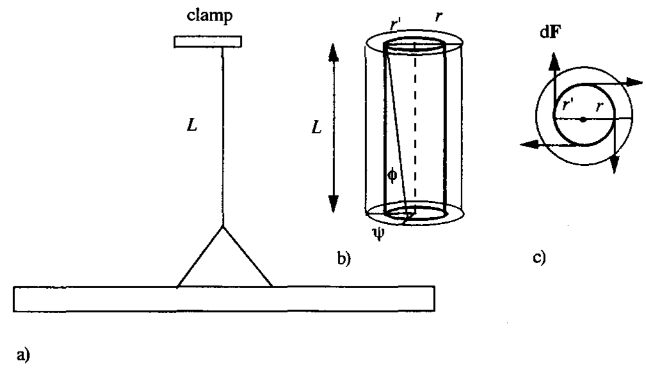

Fig. 9. (a) Torsional pendulum. (b) Strain angle ϕ in the wire rotated through an angle ψ. (c) Forces on an infinitesimal annulus of radius r' of the wire.

wire gives a smooth axis for the rod to rotate around and it prevents torsional oscillations that tend to tangle string. The upper ends of the wire are attached to short sections of string that are firmly clamped to a rigid support.

To obtain chaotic motion, the rod must be capable of freely rotating about its axis of support. Therefore, the wire should pass through the hole closest to the center of mass. Then, when the upper arm of the pendulum is released from a large angle, the rod will rotate about its axis, sometimes in one direction, sometimes the reverse. It is the unpredictability of these reversals that makes this motion chaotic.

The equations of motion for the system are found from its Lagrangian, Eq. (27). Numerical integration will yield chaotic solutions similar to those obtained experimentally.[7,9]

V. TORSIONAL PENDULUM

In a true torsional pendulum, the restoring torque comes from the angular strain ϕ in a suspending filament (Fig. 9). This angle is related to the shear stress S by[10]

$$S=G\phi,$$

where G is the modulus of rigidity (shear modulus). From Fig. 9(b) we see that for small angles, ϕ is related to the angle of rotation ψ of the wire by

$$\phi=(r'/L)\psi$$

so the torque on an infinitesimal annulus of the wire [Fig. 9(c)] is

$$d\tau=|\mathbf{r}'\times d\mathbf{F}|=r'SdA=2\pi(G/L)\psi r'^3dr'.$$

Integrating over the cross-sectional area of the wire, we find the standard expression for the torque in a twisted wire of circular cross section[10]

$$\tau=-\frac{G\pi r^4}{2L}\psi, \tag{31}$$

where r is the radius of the wire and L is its length. The period of the torsional pendulum is

$$T_0=2\pi\sqrt{\frac{I}{G\pi r^4/2L}}=2\pi\sqrt{\frac{2LI}{G\pi r^4}} \tag{32}$$

which for the rod supported horizontally is

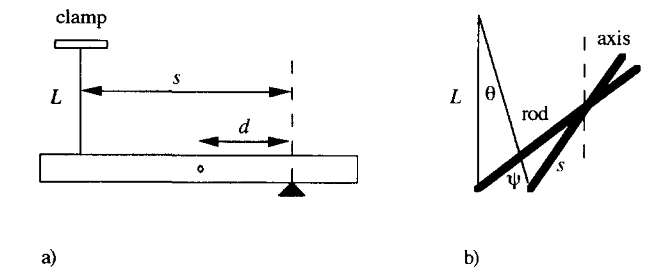

Fig. 10. (a) The horizontal pendulum. (b) Perspective view of the rod rotated out of the equilibrium plane of the system. The rod is horizontal and is rotated through an angle Ψ about the vertical axis through the pivot. For $\Psi \ll 1$, the string makes an angle $\theta = (s/L)\Psi$ with the vertical.

$$T_0 = \sqrt{\frac{2\pi L m H^2}{3 G r^4}} \quad \text{or} \quad \frac{T_0^2}{L} = \frac{2\pi m H^2}{3 G r^4}. \tag{33}$$

For example, with 26 gauge copper ($r = 0.0225$ cm) T_0^2/L was found to be 307 s^2/m, which yields $G = 4.36 \times 10^{10}$ Pa.

For the rod suspended vertically,

$$\frac{T_0^2}{L} = \frac{4\pi m R^2}{G r^4} \tag{34}$$

which, for the same wire, was found to be 1.44 s^2/m. This yields $G = 4.29 \times 10^{10}$ Pa, in agreement with the value found when the rod was suspended horizontally.

For the vertically suspended rod, care must be taken that the rod hangs exactly vertical. If the rod makes an angle β with the vertical, the moment of inertia is given by Eq. (19) with $\alpha = 90° - \beta$. Since $I_H \approx 200 I_R$, even a very small inclination β will introduce additional terms into Eq. (34).

VI. NOVELTY OSCILLATORS

A. Horizontal pendulum

The rod is suspended at one point by a string of length L and oscillates around a vertical axis through a second point that rests on a support that is a distance d from the center of mass of the rod and a distance s from the point of attachment of the string [Fig. 10(a)]. As the rod rotates through an angle ψ about this axis, the string rotates through an angle θ [Fig. 10(b)]. For small angular displacements, these angles are related by

$$L\theta = s\psi.$$

The torque exerted by the string about the axis of rotation is

$$\tau = -F\theta s = -F s^2 \psi / L,$$

where F, the tension in the cord, is mgd/s. Thus the period of oscillation is

$$T = 2\pi \sqrt{\frac{LI}{mgds}} = 2\pi \sqrt{\frac{L(H^2/12 + d^2)}{gds}}. \tag{35}$$

There are many obvious variations on this oscillator, such as inclining the rod and angling the string.

B. Rocking oscillation

In addition to swinging harmonically about a pin through a hole, the physical pendulum [Fig. 1(a)] will rock laterally about an axis perpendicular to the pin (Fig. 11). This is not simple harmonic motion, and the analysis provides an interesting exercise in the solution of an inhomogenous differential equations.

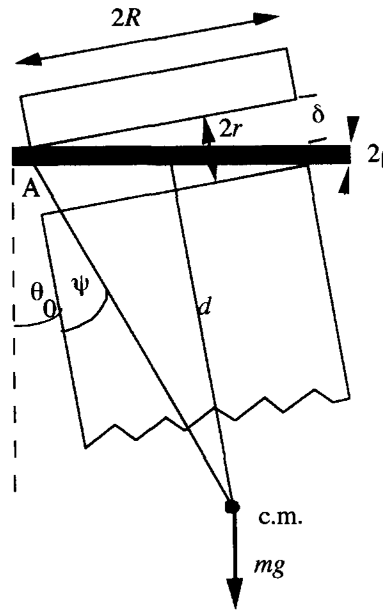

Fig. 11. The rod at its maximum angle θ_0 of lateral oscillation.

As shown in Fig. 11, the rod can be tilted laterally through a maximum angle $\theta_0 = \delta/2R$, where $\delta = 2(r - \rho)$ is the diameter of the hole minus the diameter of the pin. The torque about the axis through the upper edge of the hole (A), when the rod is tilted through any angle θ between 0 and θ_0 is

$$\tau = -mgs(\theta + \psi),$$

where $s = \sqrt{R^2 + d^2} \approx d$ is the distance from A to the center of mass of the rod and $\psi = R/d$ is a constant. For the first quarter of an oscillation, θ obeys the equation

$$\ddot{\theta} = -\omega_0^2(\theta + \psi), \tag{36}$$

where $\omega_0^2 = mgd/I$. This has the solution

$$\theta = -\frac{R}{d} + \left(\frac{\delta}{2R} + \frac{R}{d}\right) \cos \omega_0 t. \tag{37}$$

As t goes from 0 to $T/4$, θ goes from $\delta/2R$ to 0, so the period is

$$T = \frac{4}{\omega_0} \arccos\left(\frac{R/d}{\delta/2R + R/d}\right). \tag{38}$$

This period depends on the amplitude $\delta/2R$, going to zero as the amplitude goes to zero. As the rod's oscillations dampen, this increase in frequency is readily observed.

Table III shows the calculated and measured results using two different pins made from paper clips of different diameters. In each case, the time for five oscillations was repeatedly measured. The agreement with Eq. (38) is within the expected uncertainties for the thicker pin, but the measured periods are 10% smaller than the calculated periods in the

Table III. Measured and calculated periods of a rod oscillating laterally on pins of different radii ρ through holes at different distances d_0 from the center of mass (c.m.).

Distance, d^0 from c.m. to center of hole (cm)	Pin radius: 0.040 cm Period, T measured (s)	Period, T calculated (s)	Percent error	Pin radius: 0.066 cm Period, T measured (s)	Period, T calculated (s)	Percent error
10.63	0.55	0.602	8.6	0.55	0.557	1.2
6.27	0.50	0.507	1.3	0.47	0.455	−3.3
1.80	0.41	0.479	14.4	0.40	0.402	0.5

case of the thinner pin. This is because there is dampening during the first five oscillations on the thinner pin, decreasing the observed period.

C. x^3 oscillator

An oscillator subject to the force $F=-hx^3$ can be built by attaching a vertical spring to a glider on an air track.[11] By dimensional analysis, the period of such an oscillator can be shown to be

$$T=\frac{c}{A}\sqrt{\frac{m}{h}}, \tag{39}$$

where A is the amplitude of the oscillation and c is a numerical constant. By numerical analysis, c is found to be 7.416.

The period can also be found from the energy integral.[12] With the substitution, $x=A\cos\theta$, we get

$$T=4\sqrt{\frac{m}{2}}\int_0^A \frac{dx}{\sqrt{\frac{1}{4}hA^4-\frac{1}{4}hx^4}}$$
$$=4\sqrt{\frac{m}{h}\frac{1}{A}}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}=c\sqrt{\frac{m}{h}\frac{1}{A}},$$

where

$$c=4K(0.5)=7.416\ 29. \tag{40}$$

An oscillator subject to the torque $\tau=-h\psi^3$ can be built from a rigid rod that is supported by a thin filament, as in a torsional pendulum [Fig. 12(a)]. This filament, selected to give a very small restoring torque, passes through a small hole in an auxiliary rod. This hole constrains the rod to oscillate about an axis coinciding with the filament. A string, selected to be light and to hang straight under its own weight, passes vertically through an end hole of the rod. The upper end of this string is attached to a short weak spring clamped from above and the lower end is clamped without tension from below.

When the rod is rotated through an angle ψ about the supporting filament, the hole through which the second filament passes rotates a distance $x=d\psi$ [Fig. 12(b)]. The filament plus spring stretches to the length

$$L=\sqrt{L_1^2+x^2}+\sqrt{L_2^2+x^2}$$

which, to lowest order in x, produces the tension

$$F=k(L-L_1-L_2)=k(x^2/L_{\text{eff}}),$$

where k is the spring constant of the spring and L_{eff} is defined in Eq. (22).

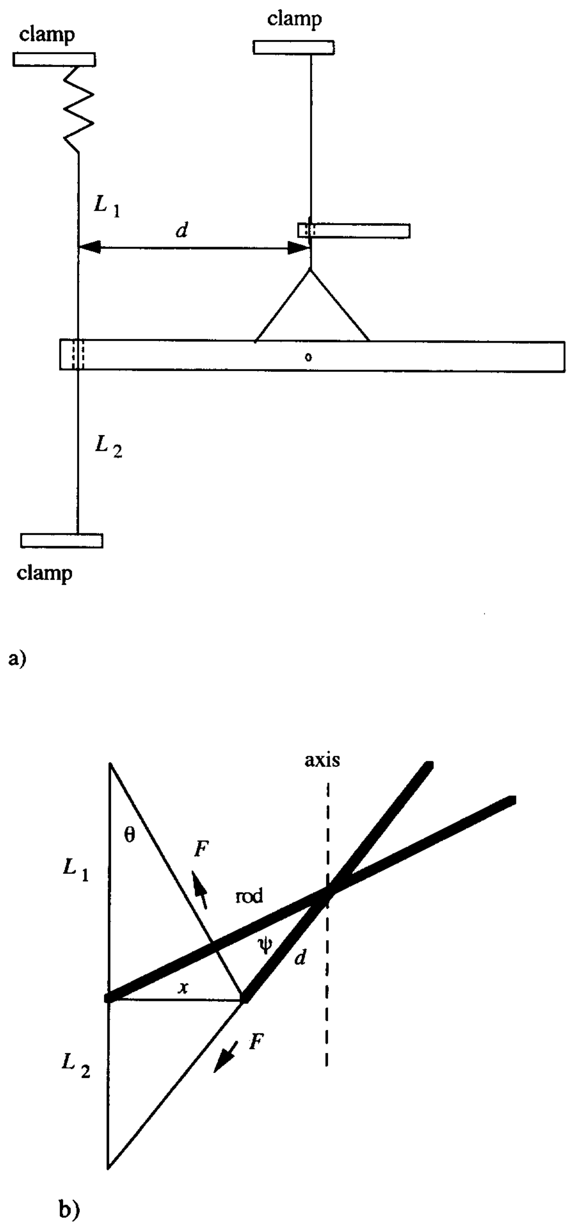

Fig. 12. (a) The x^3 oscillator. (b) When the rod rotates out of the equilibrium plane of the system, the stretched spring extends forces along the filament that exert torques about the rod's axis of rotation.

The torque produced by this force is

$$\tau = -F(dx/L_1 + dx/L_2) = -2kdx^3/L_{\text{eff}}^2$$

and the linear acceleration is

$$a = d\alpha = d\tau/I_H = -(24kd^2/mH^2L_{\text{eff}}^2)x^3. \tag{41}$$

So, by Eq. (39), the period is

$$T = 7.41\sqrt{\frac{m}{6kAd} \frac{HL_1L_2}{(L_1+L_2)}}, \tag{42}$$

where A is the linear amplitude of the oscillation.

A practical x^3 oscillator was made by suspending the rod from a 80-cm-long piece of thin monofilament (6 lb test), which had a very long torsional period (about 3 min). The end string was braided nylon fishing line, which hangs reasonably straight under its own weight. Its upper end was tied to a short spring ($k = 14$ N/m) attached to a stand, and its lower end was clamped in a small vise that sat on a scissor jack. The jack was adjusted up and down until the product TA of the period of the oscillation and the amplitude was constant. At this point, it was assumed that the tension in the string was zero and the string had no slack.

With $k = 14$ N/m, $L_1 = 31$ cm, and $L_2 = 26$ cm, the product TA of the period T and the amplitude $A(= d\theta)$ given by Eq. (42) is 11.2 m s. This agrees with measurements to within the experimental uncertainty of 5%.

VII. CONCLUSIONS

A surprisingly large number of experiments can be done with a rigid rod that involve concepts and theoretical methods from all levels of the mechanics curriculum. With careful analysis, 0.5% agreement between experiment and theory is obtained in many cases. This enables one to design student experiments that involve some discovery. At the most elementary level, freshman engineering-technology students at Northeastern University measure the period of the physical pendulum for oscillations about axes at different distances from the center of mass[13] (Sec. II A). They are not told about the period going through a minimum, but are expected to find such an effect experimentally and, ideally, to see that it follows from the formula for the period.

Intermediate students might be asked to derive the formula for the period of the horizontal pendulum (Sec. VI A). As they develop different theories, they can test them experimentally, refining them as needed. Experimental work is an essential part of the analytic process, and should be incorporated into intermediate mechanics courses. (I got few of my derivations right the first time.) Even at the advanced level, students should test their solutions to coupled oscillations problems with measurements on real systems. The normal modes of the unequal-arm bifilar pendulum suspended at its radii of gyration (Sec. III D) provide dramatic demonstrations of the power of analysis to predict phenomena.

The chaotic double pendulum (Sec. IV B) is inexpensive enough so that all teachers can have one in their bag of tricks. It demonstrates how even simple systems can at times behave with surprising complexity.

APPENDIX: EFFECT OF FINITE PIN RADIUS

We consider a rigid body supported on a pin of radius ρ passing through a hole of radius r [Fig. 13(a)]. As the body rotates through an angle θ, we assume it rolls without slipping around the pin a distance $\rho\phi$, where ϕ is angular displacement around the pin of the point of contact. This distance is also equal to $r\psi$, where ψ is the angular displacement around the hole of the point of contact. Thus we have

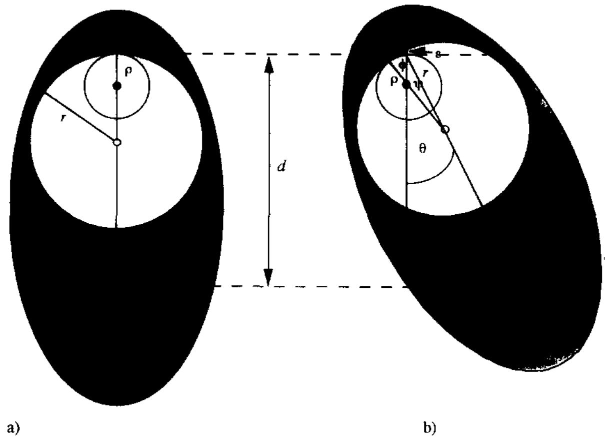

Fig. 13. (a) A physical pendulum in its equilibrium position. (b) As the pendulum rotates through an angle θ with respect to the vertical, it rolls without slipping through an angle ϕ about the pin.

$$\rho\phi = r\psi.$$

But from Fig. 13(b) we see that ϕ is the exterior angle to a triangle in which θ and ψ are interior angles, so $\phi = \theta + \psi$, and

$$\rho\phi = r(\phi - \theta)$$

or

$$\theta = (1 - \rho/r)\phi. \tag{A1}$$

Let d be the distance from the center of mass to the top of the hole. Initially this is the same as the distance d' to the top of the pin, but as the body rotates, $d' = d - \Delta$, where Δ is the distance the top of the hole moves from the top of the pin. Relative to the top of the pin, the center of mass rises the distance $d - (d - \Delta)\cos\theta$. The torque on the body is thus

$$\tau = -\frac{dU}{d\theta} = -mgd \sin\theta - mg\frac{d(\Delta\cos\theta)}{d\theta}. \tag{A2}$$

Writing the equation of the hole in polar coordinates with origin at the center of the pin, we have

$$\rho + \epsilon = -r(1 - \rho/r)\cos\phi + r\sqrt{1 - (1 - \rho/r)^2 \sin^2\phi}.$$

Expanding this, we find to lowest order in ϕ^2

$$\epsilon = 1/2\rho(1 - \rho/r)\phi^2.$$

Table IV. Experimental and theoretical periods of a ring oscillating on pins of various diameters. The ring's inner radius is $r = 7.78$ cm and its outer radius is $r' = 8.42$ cm.

Radius of pin, ρ (cm)	Period, T measured (s)	Period, T Eq. (A5) (s)
0.040	0.804	0.806
0.635	0.765	0.775
5.08	0.489	0.476

This can be expressed in terms of θ using Eq. (A1).

$$\epsilon = 1/2\rho(1-\rho/r)^{-1}\theta^2. \tag{A3}$$

In the small angle limit, $\epsilon \approx \Delta \approx \Delta \cos\theta$, so from Eq. (44), the torque is

$$\tau = -mg[d + \rho(1-\rho/r)^{-1}]\theta. \tag{A4}$$

Thus the angular frequency of a physical pendulum that rotates on a pin of radius ρ passing through a hole of radius r (Fig. 13) is given by Eqs. (6) and (7) of Sec. II A. In the case of the rigid rod, this yields a small correction. However, for a ring rotating on a pin, it is a major effect.

The moment of inertia of a ring with inner radius r and outer radius r', about an axis perpendicular to the ring at a distance r from the center, is

$$I/m = 1/2(r^2 + r'^2) + r^2$$

so its period of oscillation on a pin of radius ρ is

$$T = 2\pi \sqrt{\frac{1/2(r^2 + r'^2) + r^2}{g[r + \rho/(1-\rho/r)]}}. \tag{A5}$$

This reduces to the textbook formula $2\pi\sqrt{2r/g}$ for an infinitely thin ring on an infinitely thin pin (knife edge) and agrees with the period given by the exact expression for the acceleration of a ring on a pin.[14,15]

Table IV gives comparisons of theory and experiments for an aluminum ring of inner radius $r = 7.78$ cm oscillating on pins of various diameters.

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Periodic orbits of the integrable swinging Atwood's machine

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We identify *all* the periodic orbits of the integrable swinging Atwood's machine by calculating the rotation number of each orbit on its invariant tori in phase space, and also providing explicit formulas for the initial conditions needed to generate each orbit. © *1995 American Association of Physics Teachers.*

I. INTRODUCTION

Integrable Hamiltonian systems typically display an infinity of distinct periodic and quasiperiodic orbits. The bounded Kepler problem often studied in classical mechanics is not typical in this respect since it only exhibits periodic orbits of a simple type. This is because the Kepler problem has a third invariant, the Runge–Lenz vector,[1] in addition to the energy and angular momentum. This third invariant forces all the bounded motions to be periodic. A somewhat more typical example of an integrable two-degree of freedom Hamiltonian system is the swinging Atwood's machine when the mass ratio ($\mu = M/m$) of the nonswinging to swinging mass is equal to three.[2,3]

Numerical studies of the integrable swinging Atwood's machine exhibit a plethora of distinct periodic and quasiperiodic orbits, and when viewed in configuration space it appears to be a difficult task to organize and classify all these different types of orbits.