Assignment 1: Quantum Field Theory
Due Date: 6 Feb. 2018, 10 am

1. Show that $x^\mu y_\mu$ is Lorentz invariant, where x and y are 4-vectors.

2. Show that the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2}m\phi_1^2 - \frac{1}{2}m\phi_2^2 + g(\phi_1^2 + \phi_2^2)^2$$

   is invariant under the SO(2) transformation,

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.$$

3. In this question, we attempt to find the equation of motion of a particle inside a constant scalar field $\phi(x)$. The Lagrangian is

$$\mathcal{L} = -\big(m + \phi(x)\big)\sqrt{1 - \dot{x}^2}.$$

   This is a $(1+1)$ dimensional classical field. Write down the equation of motion.

4. The matrix Λ represents the Lorentz transformation of an upstairs (contravariant) vector whereas M represents the Lorentz transform of a downstairs (covariant) vector, i.e.

$$\begin{aligned} (A')^\mu &= \Lambda^\mu{}_\nu A^\nu \\ \text{and} \quad (A')_\mu &= M_\mu{}^\nu A_\nu \end{aligned}$$

   What is the relationship between Λ and M?

5. Find an equation of motion for the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2}m^2\phi^2 - \sum_{n=1}^{\infty} \lambda_n \phi^{2n+2}.$$

6. Show that the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_1)^2 + \frac{1}{2}(\partial_\mu \phi_2)^2 - \frac{1}{2}m^2\phi_2^2 - g(\phi_1^2 + \phi_2^2)^2$$

can be simplified when cast in terms of the complex scalar field

$$\begin{aligned} \psi &= \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \\ \text{and} \quad \psi^\dagger &= \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2). \end{aligned}$$

The simplified form is

$$\mathcal{L} = (\partial^\mu \psi^\dagger)(\partial_\mu \psi) - m^2 \psi^\dagger \psi - g'(\psi^\dagger \psi)^2$$

upto some scale factor.

7. The Lagrangian of a free particle is given by:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2.$$

   (a) Find the canonical momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$$

   (b) Determine the Hamiltonian density.

   (c) Defining

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)},$$

   what is Π°?