Assignment 10: Quantum Field Theory
Due Date: 2 May. 10 am

1. When a fermion is at rest, the Dirac spinor for the particle is

$$u(p^0) = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}.$$

The left and right handed components are equal.

(a) Show that $(\gamma^0 - 1)u(p^0) = 0$. [3 **Marks**]

(b) Show that $e^{+i\mathbf{K}\cdot\phi}\gamma^0 e^{-i\mathbf{K}\cdot\phi} = \frac{p}{m}$. [7 **Marks**]

2. (a) Prove that $u^\dagger(p)u(p) = 2E_\mathbf{p}\xi^\dagger\xi$. where

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi \\ \sqrt{p\cdot\bar{\sigma}}\ \xi \end{pmatrix}$$

is the momentum space solution of the Dirac equation. [5 **Marks**]

(b) Show that $\bar{u}(p)u(p) = 2m\xi^\dagger\xi$, where $\bar{u}(p) = u^\dagger(p)\gamma^0$. [5 **Marks**]

3. The momentum space solution of the Dirac equation for the antiparticle is

$$\nu^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \eta \\ -\sqrt{p\cdot\bar{\sigma}}\ \eta \end{pmatrix}$$

where $\sigma = (I, \sigma)$, $\bar{\sigma} = (I, -\sigma)$ and s labels the spin state $s = 1, 2$. Prove the identity,

$$\sum_{s=1}^{2} \nu^s(p)\bar{\nu}^s(p) = \gamma\cdot p - m$$

where $\bar{\nu}^s(p) = \nu^\dagger(p)\gamma^0$. [7 **Marks**]

4. The Dirac equation is given by

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0.$$

Find the adjoint ("barred" version) of the Dirac equation which employs $\bar{\psi}$ instead of ψ. [8 **Marks**]

5. Derive explicit representation of the boost matrix **K** and show by explicit calculations that $[K^1, K^2] = -iJ^3$. Your starting point are the representations $D(\theta^i)$ and $D(\phi^i)$. Assume we are dealing with vectors (spin= 1). [5 **Marks**]

6. I define a boost along the x axis as

$$\Lambda^\mu_\nu(x^1) = \begin{pmatrix} \gamma^1 & \gamma^1 v^1 & 0 & 0 \\ \gamma^1 v^1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Boosts along other axes are defined correspondingly.

(a) For infinitesimal boosts, construct the boost matrix $\Lambda^\mu_\nu(x^1, x^2, x^3)$. [3 **Marks**]

(b) Construct the infinitesimal rotation matrix $\Lambda^\mu_\nu(\theta^1, \theta^2, \theta^3)$. [3 **Marks**]

(c) Construct the infinitesimal Lorentz matrix $\Lambda^\mu_\nu(x^1, x^2, x^3; \theta^1, \theta^2, \theta^3)$ and show that it can be written as $\Lambda = 1 + \omega$. What is ω? [3 **Marks**]

(d) Show that $\omega^{\mu\nu} = \omega^\mu_\lambda g^{\lambda\nu}$ and $\omega_{\mu\nu} = g_{\mu\lambda}\omega^\lambda_\nu$ are antisymmetric. [3 **Marks**]

(e) What is the relationship between θ^i, v^i and the terms of ω? [3 **Marks**]