Assignment 3: Quantum Field Theory
Due Date: 19 Feb. 2018, 4 pm

1. Prove the Bianchi identity,

$$\partial_\mu F_{\nu\tau} + \partial_\nu F_{\tau\mu} + \partial_\tau F_{\mu\nu} = 0,$$

and show that the homogeneous Maxwell equations can be derived from it.

2. Under what conditions will the time derivative of Π^μ (the canonical momentum conjugate to the field ϕ) will be zero?

3. The Lagrangian is given by,

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2$$

   (a) Find the energy-momentum tensor $T^{\mu\nu} = \Pi^\mu \partial^\nu \phi - g^{\mu\nu}\mathscr{L}$.

   (b) Find T^{00} and verify that it is equal to the Hamiltonian density.

   (c) Find the 4-divergence of $T^{\mu\nu}$. Is the energy-momentum current conserved?

   (d) Find the total energy and

   (e) momentum inside the field.

4. Consider the source-free electromagnetic field,

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}.$$

   (a) Compute the canonical momentum conjugate to A_β defined as,

$$\Pi^{\alpha\beta} = \frac{\partial \mathscr{L}}{\partial(\partial_\alpha A^\beta)} = \frac{\partial \mathscr{L}}{\partial(A_{\beta,\alpha})}.$$

   (b) Compute the energy-momentum tensor:

$$\begin{aligned} T^{\mu\nu} &= \Pi^\mu \partial^\nu \phi - g^{\mu\nu}\mathscr{L} \\ &= \Pi^{\mu\beta}\partial^\nu A_\beta - g^{\mu\nu}\mathscr{L}, \end{aligned}$$

   where the second index on Π is introduced for the field A_β.

   (c) Is $T^{\mu\nu}$ symmetric?

(d) Add $\partial_\lambda X^{\lambda\mu\nu}$, where $X^{\lambda\mu\nu} = F^{\mu\lambda} A^\nu$ to $T^{\mu\nu}$ to construct $\bar{T}^{\mu\nu}$. Is $\bar{T}^{\mu\nu}$ symmetric? Show that $X^{\lambda\mu\nu} = -X^{\mu\lambda\nu}$.

(e) Find $\bar{T}^{00}$ and the conserved charge P^0.

(f) Find $\bar{T}^{0i}$ and the conserved charge P^i.

(g) Physically interpret P^μ.

(h) Could you construct a continuity equation for $T^{\mu\nu}$?