Assignment 4: Quantum Field Theory Due Date: 26 Feb. 4 pm

1. Using the definition of the Fourier transform of $\hat{x}_k$ and the decomposition of $\hat{x}_k$ in terms of the creation and annihilation operators, derive the following mode expansion,

$$\hat{x}_j = \sqrt{\frac{\hbar}{2m\omega N}} \sum_{\mathbf{k}} \left(\hat{a}_{\mathbf{k}}\, e^{+ikja} + \hat{a}^{\dagger}_{\mathbf{k}}\, e^{-ikja} \right) \tag{1}$$

2. Show that $[\hat{a}_{\mathbf{k}}, \hat{a}^{\dagger}_{\mathbf{k}}] = 1$. Your starting point is the position-momentum uncertainty.

3. Consider the two-particle momentum state $|\mathbf{pq}\rangle$.

   (a) Show that

$$\langle \mathbf{p}'\mathbf{q}'|\mathbf{pq}\rangle = \delta^{(3)}(\mathbf{p}' - \mathbf{q})\, \delta^{(3)}(\mathbf{q}' - \mathbf{p}) \pm \delta^{(3)}(\mathbf{p}' - \mathbf{p})\, \delta^{(3)}(\mathbf{q}' - \mathbf{q}). \tag{2}$$

   (b) Using this result show that

$$\frac{1}{\sqrt{2}} \int d^3p'\, d^3q'\, \phi_{\mathbf{p}'}(\mathbf{x})\, \phi_{\mathbf{q}'}(\mathbf{y})\, \langle \mathbf{p}'\mathbf{q}'|\mathbf{pq}\rangle = \frac{1}{\sqrt{2}} \left(\phi_{\mathbf{q}}(\mathbf{x})\phi_{\mathbf{p}}(\mathbf{y}) \pm \phi_{\mathbf{p}}(\mathbf{x})\, \phi_{\mathbf{q}}(\mathbf{y}) \right) \tag{3}$$

   where $\phi_p(\mathbf{x}) = \langle \mathbf{x}|\mathbf{p}\rangle$.

4. Given the Hamiltonian for N coupled oscillators on a ring,

$$\hat{H} = \sum_j \left(\frac{\hat{p}_j^2}{2m} + \frac{1}{2} K \left(\hat{x}_{j+1}^2 - \hat{x}_{j+1}^2 \right) \right), \tag{4}$$

   derive the $\mathbf{k}$ space Hamiltonian. Show the *complete* working at all steps.

5. For a 3D harmonic oscillator the Hamiltonian is

$$\hat{H} = \hbar\omega \sum_{i=1,2,3} \left(\hat{a}_i^{\dagger}\hat{a}_i + \frac{1}{2} \right). \tag{5}$$

   With the transformations,

$$\hat{b}_3^{\dagger} = \hat{a}_3^{\dagger} \tag{6}$$

$$\hat{b}_1^{\dagger} = -\frac{1}{\sqrt{2}}(\hat{a}_1^{\dagger} + i\hat{a}_2^{\dagger}) \tag{7}$$

$$\hat{b}_2^{\dagger} = \frac{1}{\sqrt{2}}(\hat{a}_1^{\dagger} - i\hat{a}_2^{\dagger}) \tag{8}$$

   show that

$$\hat{H} = \hbar\omega \sum_{i=1,2,3} \left(\hat{b}_i^{\dagger}\hat{b}_i \right). \tag{9}$$

6. Derive the commutation relationship for the fermionic field operators (in space).