Assignment 5: Quantum Field Theory
Due Date: 5 March 2018, 4 pm

1. The second-quantization version of a single particle operator is:

$$\hat{A} = \sum_{\alpha,\,\beta} \langle \alpha | \hat{A} | \beta \rangle \hat{a}^\dagger_\alpha \hat{a}_\beta.$$

If $|\alpha\rangle$ and $|\beta\rangle$ are momentum eigenstates, show that this operator in the position space is given by

$$\hat{A} = \int d^3x \; A(\mathbf{x}) \; \hat{\psi}^\dagger(\mathbf{x}) \; \hat{\psi}(\mathbf{x}).$$

I am looking for a neat and systematic solution.

2. (a) Determine $\left\{\hat{\psi}(\mathbf{x}),\, \hat{c}^\dagger_{\mathbf{p}}\right\}$.

   (b) Consider the fermionic particles in states $\mathbf{p}$ and $\mathbf{q}$ generated by the operators,

$$\hat{c}^\dagger_{\mathbf{q}} \hat{c}^\dagger_{\mathbf{p}} |\, 0\rangle = |\, \mathbf{pq}\rangle.$$

   Find the representation of $|\,\mathbf{pq}\rangle$ in the position basis. For this purpose, you will calculate $\langle \mathbf{x}_1 \mathbf{x}_2 |\, \mathbf{pq}\rangle$ while using the (anti) commutator result derived in part (a). [Hint: $|x\rangle = \hat{\psi}^\dagger(\mathbf{x}) |\, 0\rangle$].

3. (a) Consider the interaction between two particles described by

$$V(\mathbf{x}) = A\delta^{(3)}(\mathbf{x}),$$

   where A is a constant. Find the second-quantization form of the interaction.

   (b) If the inter-particle interaction is described by the Yukawa potential

$$V(r) = \frac{Ae^{-\lambda r}}{r},$$

   determine the second-quantization form of the interaction. Convince yourself that the coulomb interaction is the long range form of the Yukawa interaction, where $A = \dfrac{q^2}{2\pi\varepsilon_0}$.

4. Consider two fermions a_1 and a_2.

(a) Show that Bogoliubov transformation

$$\begin{aligned}\hat{c}_1 &= u\hat{a}_1 + v\hat{a}_2^\dagger \\ \hat{c}_2^\dagger &= -v\hat{a}_1 + u\hat{a}_2^\dagger,\end{aligned}$$

where u and v are real, preserves the canonical anticommutation relations if $u^2 + v^2 = 1$

(b) Use this result to show that the Hamiltonian

$$H = \varepsilon(\hat{a}_1^\dagger\ \hat{a}_1 - \hat{a}_2\hat{a}_2^\dagger) + \Delta(\hat{a}_1^\dagger\ \hat{a}_2^\dagger + \hat{a}_1\hat{a}_2),$$

can be diagonalized in the form

$$H = \sqrt{\varepsilon^2 + \Delta^2}(\hat{c}_1^\dagger\ \hat{c}_1 + \hat{c}_2^\dagger\ \hat{c}_2 - 1).$$

(c) What is the ground-state energy of this Hamiltonian?

(d) Write out the ground-state wavefunction in terms of the original operators $\hat{c}_1^\dagger$ and $\hat{c}_2^\dagger$ and their corresponding vacuum $|\,0\rangle, i.e.,\quad (\hat{c}_{1,2}|\,0\rangle = 0)$.

5. Defining the density matrix for a single particle as,

$$\hat{\rho}_1(\mathbf{x} - \mathbf{y}) = \langle\hat{\psi}^\dagger(\mathbf{x})\hat{\psi}(\mathbf{y})\rangle.$$

Express this matrix in the form of creation and annihilation operators.