Assignment 7: Quantum Field Theory
Due Date: 2 April, 10 am

1. In HW 3, Q4(d), we introduced the symmetric energy-momentum tensor for a source-free electromagnetic field

$$\overline{T}^{\mu\nu} = -F^{\mu\beta}\partial^\nu A_\beta + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \partial_\lambda X^{\lambda\mu\nu} \tag{1}$$

where $X^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu$. Show that this symmetrized tensor is gauge invariant. [10 marks]

2. (a) For the source-free Proca Lagrangian (corresponding to a massive vector field)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu \tag{2}$$

the energy-momentum tensor is given by,

$$\overline{T}^{\mu\nu} = -F^{\mu\beta}\partial^\nu A_\beta + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \partial_\lambda(F^{\mu\lambda}A^\nu) - \frac{1}{2}m^2 g^{\mu\nu}A_\alpha A^\alpha. \tag{3}$$

Compute the energy density in terms of **E**, **B**, A^0 and **A**. [10 marks]

(b) Now determine the momentum density in terms of these vectors. [10 marks]

3. Consider an undamped harmonic oscillator. It's Green's function is defined as

$$\left(m\frac{\partial^2}{\partial t^2} + mE_o^2\right)G(t,u) = \delta(t-u). \tag{4}$$

Use the following definitions of Fourier transformation and its inverse:

$$\begin{aligned}\tilde{F}(E) &= \int_{-\infty}^{\infty} dt f(t)e^{iEt} \\ f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dE\tilde{F}(E)e^{-iEt}.\end{aligned}$$

(a) Fourier transform Equation (4) above to obtain $\tilde{G}(E,u)$. [3 marks]

(b) Inverse Fourier transform $\tilde{G}(E,u)$ and hence determine the retarded Green function $G(t,u)$. **Notes**: (i) Sketch the function. (ii) You will need to perform contour integration in the complex plane, employing Cauchy's theorem and the reside theorem. Furthermore, you will ensure causality by adding infinitesimal imaginary components to the energy E. Clearly show all your working. [17 marks]