Assignment 8: Quantum Field Theory
Due Date: 10 April. 10 am

1. Show a neat derivation of the Heaviside function expanded as an integral,

$$\Theta(t) = \frac{i}{2\pi}\int_{-\infty}^{\infty} dz\ \frac{e^{-izt}}{z+i0^+}.$$

[10 **marks**]

2. Is

$$\Delta(x,y) = \int \frac{d^4p}{(2\pi)^4}\ \frac{i}{p^2-m^2+i0^+}e^{-ip\cdot(x-y)}$$

the Green function for the Klein-Gorden operator? Show your working. [5 **marks**]

3. In class, we derived the following expression for the free propagation for scalar fields:

$$\Delta(x,y) = \int \frac{d^4p}{(2\pi)^4}\ e^{-ip\cdot(x-y)}\ \frac{i}{(p^0)^2-E_{\mathbf{p}}^2}$$

Replace $E_{\mathbf{p}}$ with $E_{\mathbf{p}} - i0^+$, where 0^+ is an infinitesimal positive number. Compute the integral given above using the rules of contour integration, verifying in the process, that

$$\Delta(x,y) = \int \frac{d^3p}{(2\pi)^3}\Big(\Theta(x^0-y^0)e^{-ip\cdot x} + \Theta(y^0-x^0)e^{+ip\cdot x}\Big).$$

What is the role of the minute term $i0^+$? [15 **marks**]

4. (a) For a massive scalar field, the Lagrangian density is:

$$\mathscr{L} = \frac{1}{2}\Big(\partial_\mu\phi(x)\Big)^2 - \frac{1}{2}m^2\Big(\phi(x)\Big)^2.$$

Express the action $S = \int d^4x\ L$ in the momentum space and comment how S depends on the momentum representation of the Feynmann propagator for the scalar field. [10 **marks**]

(b) Now let's discretize the scalar field by positioning it on an equally spaced chain in one dimension. The discretization is achieved by:

$$\phi_j(t) = \frac{1}{\sqrt{L}}\sum_p \int \frac{d\omega}{2\pi}\ \widetilde{\phi}_p\ e^{-i(\omega t - pja)}$$

where $L = Na$ is the length of the chain, j is the spatial index and $\widetilde{\phi}_p(\omega)$ is the Fourier transform of $\phi_j(t)$. Using

$$\frac{1}{L}\sum_j e^{+i(p+q)ja} = \delta^{(1)}(p+q)$$

and the fact that we have dealing with the $(1+1)$ Minkonski space, derive the action S in momentum space.

Using results from the previous question, state the Green Function for this one-dimensional chain. The excitations of such a field are called phonons.

[15 **marks**]

5. Show that *only* if $\hat{H}_{1I}(t)$ is self-commuting at all times, does

$$\hat{U}(t_2, t_1) = e^{-i\int_{t_1}^{t_2} \hat{H}_{1I}(\tau)\ d\tau}$$

represent a solution of

$$i\frac{d}{dt_2}\hat{U}(t_2, t_1) = \hat{H}_{1I}\ \hat{U}(t_2, t_1).$$

[10 **marks**]