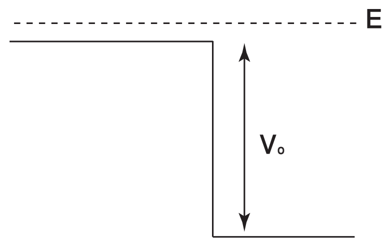
Solution Assignment 5: Modern Physics

1. Answer

In the refracting medium, the electric potential is positive and hence the potential energy seen by the electron, V, is lower. This means that the difference $E - V$ is larger, k is larger, and hence the speed is slower, $v_2 < v_1$. Hence from Bethe's law, $\sin\alpha < \sin\beta$, $\alpha < \beta$. The electron beam bends away from the normal. Hence option (c) is the correct answer.

2. Answer

One need to think carefully about this. Consider the accompanying figure.



The potential depression V_0 is large. Let's find the reflection probability R. In region I,

$$\psi_I(x) = Ae^{ik_1x} + Be^{-ik_1x},$$

and for region II,

$$\psi_{II}(x) = Ce^{ik_2x},$$

where $k_1^2 = \frac{2mE}{\hbar}$, and $k_2 = \frac{2m(E+V_0)}{\hbar}$. At the point of the precipice, $x = 0$, $\psi_I(0) = \psi_{II}(0)$ and $\psi_I'(0) = \psi_{II}'(0)$. So $A + B = C$ and $ik_1(A - B) = ik_2C$. Eliminating C from these equations,

$$\begin{aligned}
A + B &= \frac{ik_1(A - B)}{ik_2} \\
&= \frac{k_1}{k_2}A - \frac{k_1}{k_2}B \\
B\left(1 + \frac{k_1}{k_2}\right) &= A\left(\frac{k_1}{k_2} - 1\right) \\
\frac{B}{A} &= \frac{k_1 - k_2}{k_1 + k_2} \\
R &= \frac{|B|^2}{|A|^2}\frac{k_1}{k_1} \\
&= \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2.
\end{aligned}$$

If V_0 is very large, $k_2 \gg k_1$, R becomes $\left(-\frac{k_2}{k_2}\right)^2 = 1$. Since $R \simeq 1$, $T = 0$. There is zero probability for the particle to "fall over the edge" and enter region II. Hence option (a) is the correct answer.

3. Answer

Since the length of infinite well is very large i.e. 30 cm, for a small amount of energy 1 eV, the number of nodes will be very large i.e. 4.9×10^8. Since the number of nodes is very large, the waves are “squeezed" close together, the de Broglie wavelength is extremely small obscuring chances of observing the quantum wave behavior at the classical macroscopic scale. At such a high value of n, quantum effects are not visible. Another way of looking at this is that the wave function is such that the probability of finding the electron becomes equal everywhere, i.e. it imparting the electron a continuous quality rather than quantized. All of this ties in well with Bohr's corresponding principle.

4. Answer

According to uncertainty principle,

$$\Delta p \Delta x \geq \frac{\hbar}{2}.$$

If particle moves in a circle of radius r and angular momentum L, then

$$\begin{aligned} L &= pr \\ \Rightarrow \Delta L &= \Delta pr \\ \Rightarrow \Delta p &= \frac{\Delta L}{r} \\ \text{and} \qquad \Delta x &= r\Delta\theta. \end{aligned}$$

Using these values of Δp and Δx in uncertainty relation,

$$\begin{aligned} \frac{\Delta L}{r} \cdot r\Delta\theta &\geq \frac{\hbar}{2} \\ \Delta L \Delta\theta &\geq \frac{\hbar}{2}. \end{aligned}$$

Hence option (a) is the correct answer.

5. Answer

We are given that,

$$\begin{aligned} \text{Tip-sample distance} &= \alpha = 1 \text{ nm}^{-1} = 1 \times 10^9 \text{ m}^{-1} \\ \text{Distance covered by tip} &= \Delta L = 0.1 \text{ nm} = 0.1 \times 10^{-9} \text{ m}. \end{aligned}$$

Tunneling probability is,

$$T_i = e^{-2\alpha L}.$$

If tip moves closer to the surface by ΔL, final tunneling probability will become,

$$T_f = e^{-2\alpha(L-\Delta L)}.$$

ratio of tunneling probabilities is,

$$\begin{aligned}\frac{T_f}{T_i} &= \frac{e^{-2\alpha(L-\Delta L)}}{e^{-2\alpha L}}\\ &= e^{2\alpha\Delta L}\\ &= e^{2\times 10^9\times 0.1\times 10^{-9}}\\ &= e^{0.2}\\ &= 1.22.\end{aligned}$$

Hence there is a 22% increase in the tunneling current and the correct answer is (b).

6. **Answer**

$$\begin{aligned}\text{length of barrier} &= L = 35\,\text{fm} = 35\times 10^{-15}\,\text{m}\\ \text{height of barrier} &= V_o - E = 5\,\text{MeV}\\ &= 5\times 10^8\times 1.6\times 10^{-19}\,\text{m} = 8\times 10^{-13}\,\text{J}\\ \text{mass of alpha nucleus} &= m = 4\times 1.67\times 10^{-27}\,\text{kg} = 6.68\times 10^{-27}\,\text{kg}\\ \alpha &= \frac{\sqrt{2m(V_o-E)}}{\hbar}\\ \alpha &= 9.75\times 10^{14}\,\text{m}^{-1}\end{aligned}$$

So, tunneling probability is

$$\begin{aligned}T &= e^{-2\alpha L}\\ &= e^{-2\,\times\,9.75\,\times\,10^{14}\,\times\,35\,\times\,10^{-15}}\\ &\boxed{T = 2.29\times 10^{-30}}\end{aligned}$$

It is the probability that a single alpha particle hitting the surface will escape. Of course, this is a tiny probability and one may imagine that radioactivity will hardly be observable. However, the number of hits per second must also be taken in consideration.

$$\begin{aligned}\text{Number of hits per second} &= 5\times 10^{21}\,\text{s}^{-1}\\ \text{Prob. (escape in a day)} &= 2.29\times 10^{-30}\times 5\times 10^{21}\times 24\times 60\times 60\end{aligned}$$

Therefore,

$$\boxed{T_{\text{day}} = 9.89\times 10^{-4}}$$

It is the probability that an alpha particle hitting the surface will escape in a day.

(**Answer**)

Remark

Half-life can also be calculated as:

$$\begin{aligned}
\text{decay rate} &= 2.29 \times 10^{-30} \times 5 \times 10^{21}\,\mathrm{s}^{-1} \\
\text{half-life} &= \frac{\ln 2}{\text{decay rate}} \\
&= \frac{\ln 2}{2.29 \times 10^{-30} \times 5 \times 10^{21}} \\
\Aboxed{\text{half-life} &= 6.05 \times 10^{7}\,\mathrm{s}}
\end{aligned}$$

Now, we will plot T as a function of L. The MATLAB code for such a plot is:

```
%% Question # 06
L = (0:0.01:35)*10^-15;
m = 9.11*10^-31;
e = 1.6*10^-19;
V0E = 5*10^6*e;
hbar = 1.06e-34;
alpha = sqrt(2*m*V0E)/hbar;
T = exp(-2*alpha*L);
plot(L,T)
xlabel('Length of Barrier (L)') % x-axis label
ylabel('Tunneling Probability (T)') % y-axis label
title('Plot of T as a function of L')
L1 = 35*10^-15;
V0E1 = (1:0.01:5)*10^6*e;
alpha1 = sqrt(2*m*V0E1)/hbar;
T1 = exp(-2*alpha1*L1);
figure
plot(alpha1, T1)
xlabel('Height of Barrier (V_{0} - E)') % x-axis label
ylabel('Tunneling Probability (T)') % y-axis label
title('Plot of T as a function of V_o - E')
```

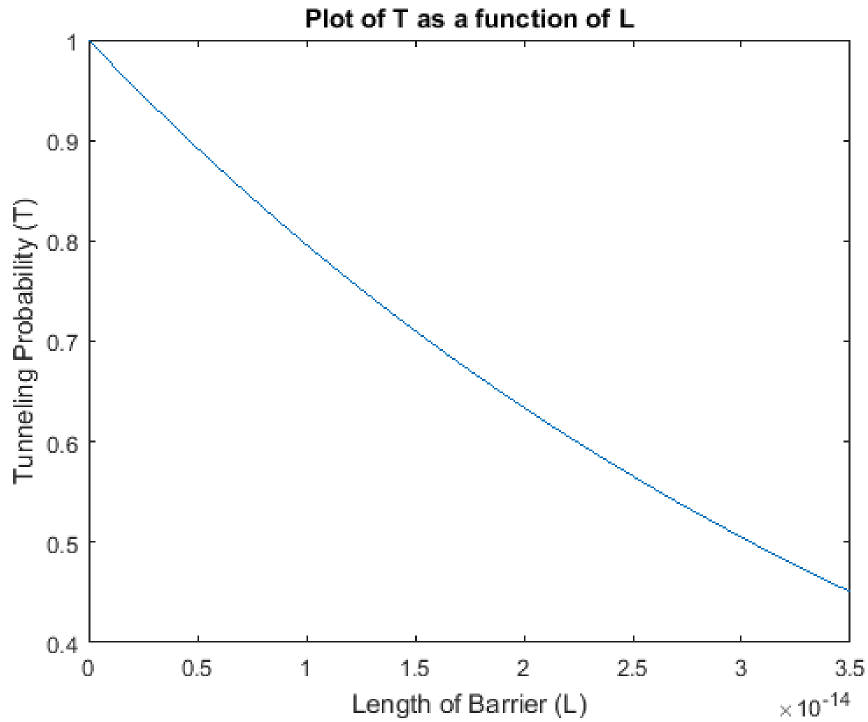


Figure 1: Plot of T as a function of L

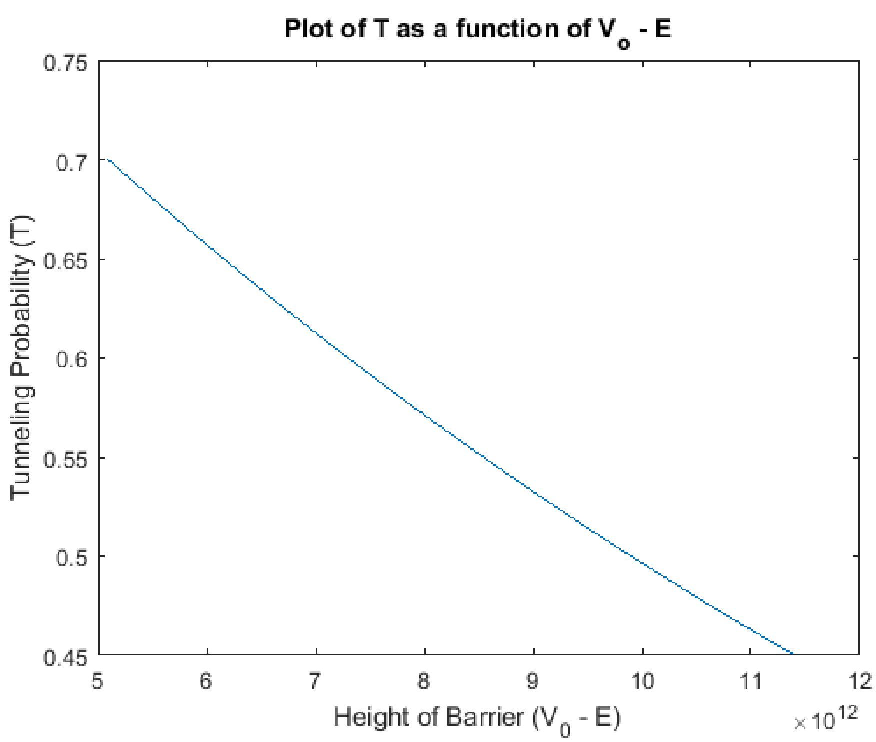


Figure 2: Plot of T as a function of $V_o - E$

7. <u>Answer</u>

(a)

It is given that

$$\psi = \frac{1}{\sqrt{2}}\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right) + \frac{1}{\sqrt{2}}\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$$

Its complex conjugate is

$$\psi^* = \frac{1}{\sqrt{2}}\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right) + \frac{1}{\sqrt{2}}\sqrt{\frac{2}{L}}\sin\left(\frac{2\pi x}{L}\right)$$

It gives

$$\int |\psi|^2 \, \mathrm{d}x = \frac{1}{L}\int_0^L \sin^2\left(\frac{\pi x}{L}\right) + \frac{1}{L}\int_0^L \sin^2\left(\frac{2\pi x}{L}\right) + \frac{2}{L}\int_0^L \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)$$
$$\int |\psi|^2 \, \mathrm{d}x = \frac{1}{L}\frac{L}{2} + \frac{1}{L}\frac{L}{2} + 0$$

So

$$\boxed{\int |\psi|^2 \, \mathrm{d}x = 1}$$

And the wave function is normalized.

(**Answer**)

(b) The states will evolve in the following way showing that the $n = 1$ state picks up a factor $e^{-i\frac{E_1}{\hbar}t}$ and the $n = 2$ state picks up a factor $e^{-i\frac{E_2}{\hbar}t} = e^{-i\frac{4E_1}{\hbar}t}$ as $E_2 = 4E_1$.

$$\Psi(x,t) = \frac{1}{\sqrt{L}}e^{-i\frac{E_1}{\hbar}t}\sin\left(\frac{\pi x}{L}\right) + \frac{1}{\sqrt{L}}e^{-i\frac{E_2}{\hbar}t}\sin\left(\frac{2\pi x}{L}\right) \tag{7.1}$$
$$\Psi(x,t) = \frac{1}{\sqrt{L}}e^{-i\frac{E_1}{\hbar}t}\left[\sin\left(\frac{\pi x}{L}\right) + e^{-i\frac{3E_1}{\hbar}t}\sin\left(\frac{2\pi x}{L}\right)\right]$$

Since $e^{-i\frac{E_1}{\hbar}t}$ is global phase, so it can be ignored.

$$\Psi(x,t) = \frac{1}{\sqrt{L}}\left[\sin\left(\frac{\pi x}{L}\right) + e^{-i\frac{3E_1}{\hbar}t}\sin\left(\frac{2\pi x}{L}\right)\right]$$

And its real part will be

$$\boxed{\mathrm{Re}\,\Psi(x,t) = \frac{1}{\sqrt{L}}\left[\sin\left(\frac{\pi x}{L}\right) + \cos\left(\frac{3E_1}{\hbar}t\right)\sin\left(\frac{2\pi x}{L}\right)\right]}$$

(**Answer**)

I take $L = 2\pi$, x goes from 0 to L, I also let $\hbar = 1$ and $E_1 = 1$ for simplicity.

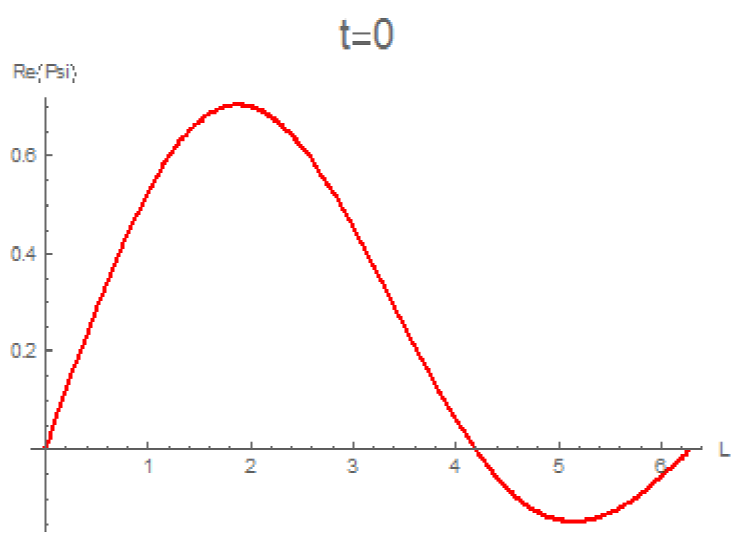
$$\mathrm{Re}\,\Psi(x,t) = \frac{1}{\sqrt{2\pi}}\left[\sin\left(\frac{x}{2}\right) + \cos(3t)\sin(x)\right]$$

The real part is periodic with time such that

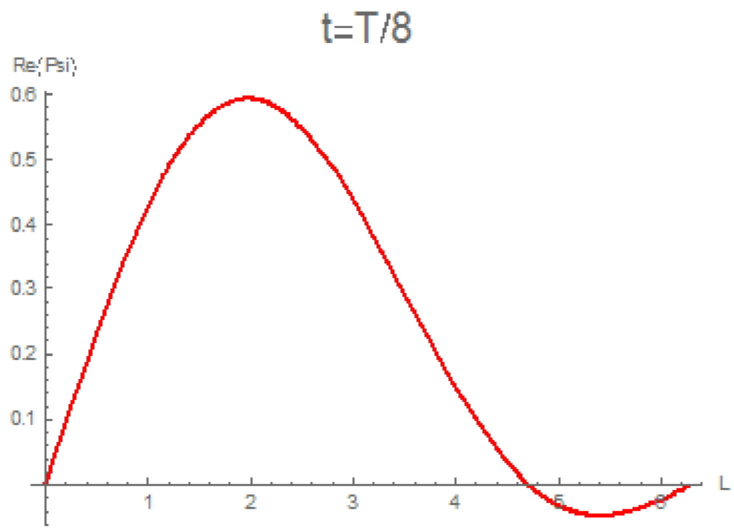
$$3T = 2\pi \qquad \Rightarrow \qquad T = \frac{2\pi}{3}$$

Using the attached *Mathematica* code, we can plot the variation of $\mathrm{Re}\,\Psi(x,t)$ at several times within the time period T. You will observe that the bigger lobe of the wavefunction changes its position from the left to right half of the well oscillating to and fro.

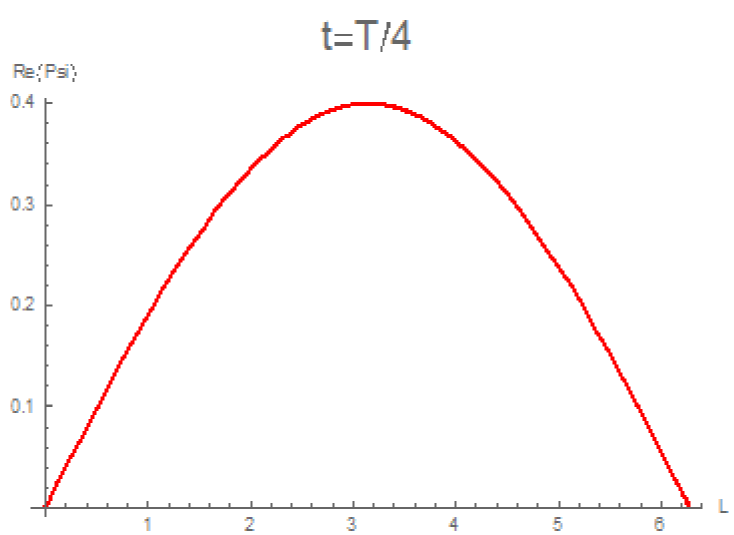
```
Psi[t_] := 1/Sqrt[2*Pi]*(Sin[x/2] + Cos[3*t]*Sin[x])
Plot[Psi[0], {x, 0, 2*Pi}, AxesLabel -> {L, Re(Psi)},
PlotLabel -> Style[t=0, 20], PlotStyle -> {Red, Thick}]
```



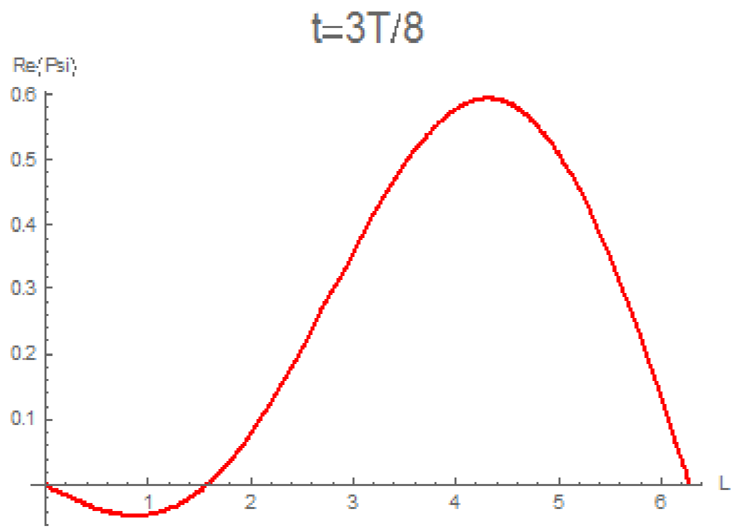

```
Plot[Psi[1/8*2*Pi/3], {x, 0, 2*Pi}, AxesLabel -> {L, Re(Psi)},
   PlotLabel -> Style[t=T/8, 20], PlotStyle -> {Red, Thick}]
```



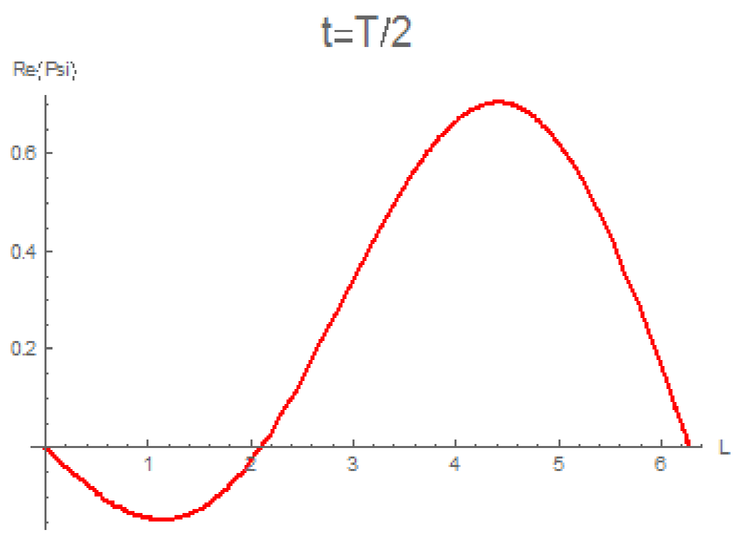

```
Plot[Psi[1/4*2*Pi/3], {x, 0, 2*Pi}, AxesLabel -> {L, Re(Psi)},
  PlotLabel -> Style[t=T/4, 20], PlotStyle -> {Red, Thick}]
```



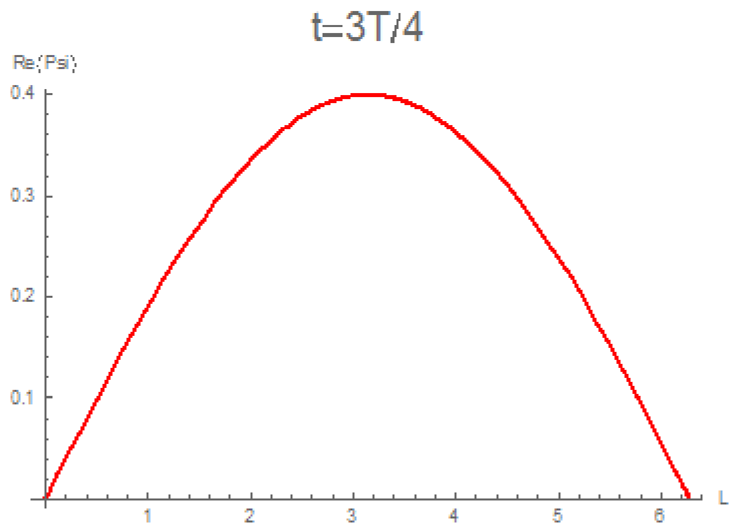

```
Plot[Psi[3/8*2*Pi/3], {x, 0, 2*Pi}, AxesLabel -> {L, Re(Psi)},
  PlotLabel -> Style[t=3T/8, 20], PlotStyle -> {Red, Thick}]
```




```
Plot[Psi[1/2*2*Pi/3], {x, 0, 2*Pi}, AxesLabel -> {L, Re(Psi)},
  PlotLabel -> Style[t=T/2, 20], PlotStyle -> {Red, Thick}]
```




```
Plot[Psi[3/4*2*Pi/3], {x, 0, 2*Pi}, AxesLabel -> {L, Re(Psi)},
  PlotLabel -> Style[t=3T/4, 20], PlotStyle -> {Red, Thick}]
```



(c) By taking complex conjugate of Eq. (7.1), we obtain

$$\Psi^*(x,t) = \frac{1}{\sqrt{L}} e^{i\frac{E_1}{\hbar}t} \sin\left(\frac{\pi x}{L}\right) + \frac{1}{\sqrt{L}} e^{i\frac{E_2}{\hbar}t} \sin\left(\frac{2\pi x}{L}\right)$$

Leading to

$$\begin{aligned}|\Psi|^2 =& \frac{1}{L}\left[\sin^2\left(\frac{\pi x}{L}\right) + \sin^2\left(\frac{2\pi x}{L}\right) + \right. \\ & \left. \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\exp\left(i\frac{E_1-E_2}{\hbar}t\right) + \sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\exp\left(-i\frac{E_1-E_2}{\hbar}t\right)\right]\end{aligned}$$

By using $\dfrac{e^{ix}+e^{-ix}}{2} = \cos x$, we obtain

$$\boxed{|\psi|^2 = \frac{1}{L}\left[\sin^2\left(\frac{\pi x}{L}\right) + \sin^2\left(\frac{2\pi x}{L}\right) + 2\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{E_1-E_2}{\hbar}t\right)\right]}$$

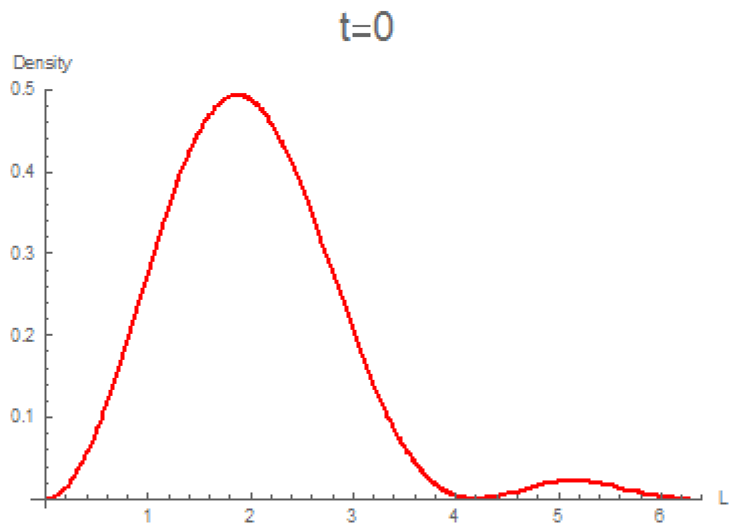
(**Answer**)

(d) As we did in part (b), we can write probability density for the same values of $L, \hbar, E_1, E_2$ as:

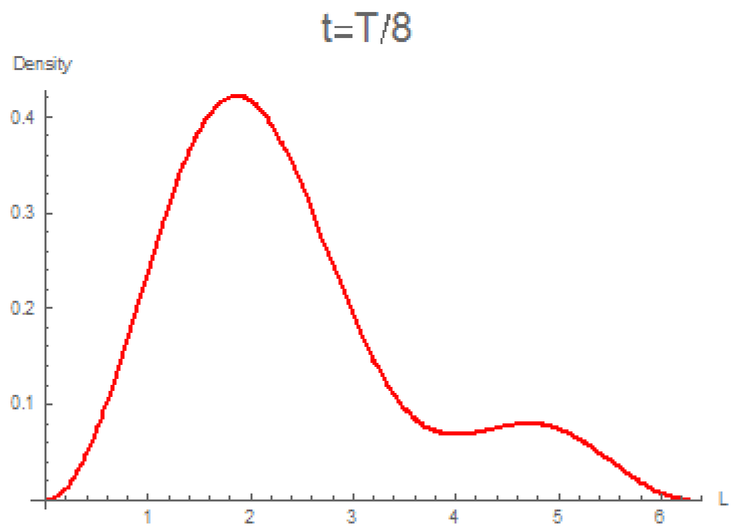
Mathematica code to plot probability density profile at different times is:

$$|\psi|^2 = \frac{1}{2\pi}\left[\sin^2\left(\frac{x}{2}\right) + \sin^2(x) + 2\sin\left(\frac{x}{2}\right)\sin(x)\cos(3t)\right]$$

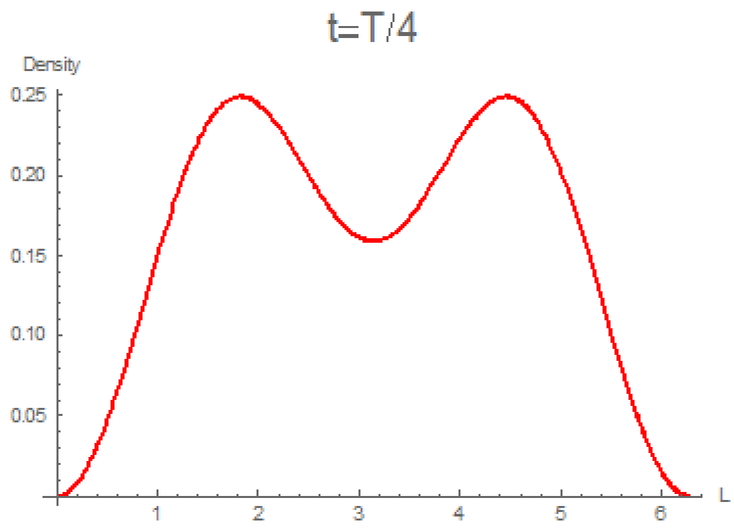
```
Pdensity[tt_] := 1/(2*Pi)*((Sin[x/2])^2 + (Sin[x])^2 + 2*Sin[x/2]*Sin
   [x]*Cos[3*tt])
Plot[Pdensity[0], {x, 0, 2*Pi}, AxesLabel -> {L, Density},
PlotLabel -> Style[t=0, 20], PlotStyle -> {Red, Thick}]
```



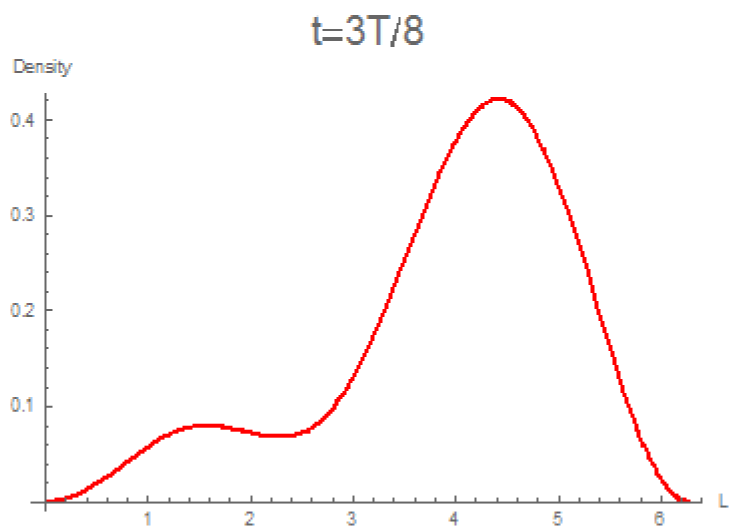

```
Plot[Pdensity[1/8*2*Pi/3], {x, 0, 2*Pi}, AxesLabel -> {L, Density},
PlotLabel -> Style[t=T/8, 20], PlotStyle -> {Red, Thick}]
```



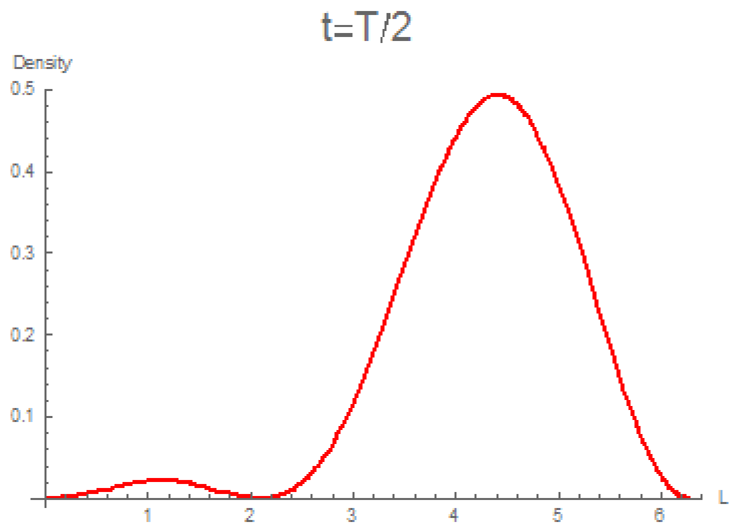

```
Plot[Pdensity[1/4*2*Pi/3], {x, 0, 2*Pi}, AxesLabel -> {L, Density},
PlotLabel -> Style[t=T/4, 20], PlotStyle -> {Red, Thick}]
```



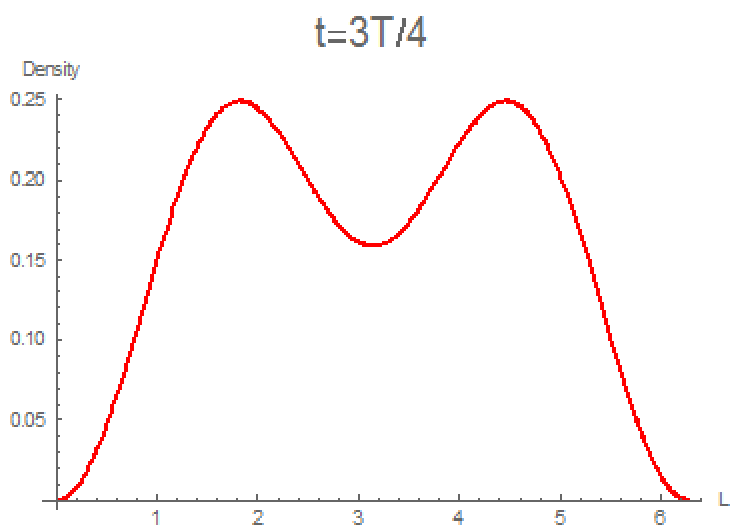

```
Plot[Pdensity[3/8*2*Pi/3], {x, 0, 2*Pi}, AxesLabel -> {L, Density},
PlotLabel -> Style[t=3T/8, 20], PlotStyle -> {Red, Thick}]
```




```
Plot[Pdensity[1/2*2*Pi/3], {x, 0, 2*Pi}, AxesLabel -> {L, Density},
PlotLabel -> Style[t=T/2, 20], PlotStyle -> {Red, Thick}]
```




```
Plot[Pdensity[3/4*2*Pi/3], {x, 0, 2*Pi}, AxesLabel -> {L, Density},
PlotLabel -> Style[t=3T/4, 20], PlotStyle -> {Red, Thick}]
```



8. **Answer**

(a) The time-independent Schrodinger wave equation is

$$-\frac{2m}{\hbar^2}\frac{\mathrm{d}^2\psi(x)}{\mathrm{d}x^2}+(V_o-E)\psi(x)=0 \tag{8.1}$$

It is obvious from Eq. (8.1) that for Region-I (left side of the figure)

$$\boxed{\psi_I(x)=0}$$

Because it is outside the well.

(**Answer**)

Let's re-write Eq. (8.1) in the following fashion

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \frac{2m}{\hbar^2}(E - V_o)\psi = 0$$

For Region-II (middle of the figure), we are inside the well, i.e. $V_o = 0$, so Eq. (8.1) becomes now

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \frac{2mE}{\hbar^2}\psi = 0 \tag{8.1}$$

By putting $k^2 = \dfrac{2mE}{\hbar^2}$, we get

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + k^2\psi = 0$$

Its general solution is

$$\psi_{II}(x) = A'e^{+ikx} + B'e^{-ikx} \tag{8.2}$$

By applying boundary condition, at $x = 0$, Eq. (8.2) yields $B' = -A'$, so

$$\psi_{II}(x) = 2iA'e^{+ikx}$$
$$\psi_{II}(x) = 2iA'\sin(kx)$$

By putting $2iA' = A$ he wave function becomes

$$\boxed{\psi_{II}(x) = A\sin(kx)}$$ (**Answer**)

For Region-III (right side of the figure), we can rewrite Eq. (8.1) as

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - \frac{2m}{\hbar^2}(V_o - E)\psi = 0$$

For a bound system, $V_o > E$ and by putting $\alpha^2 = \dfrac{2m(V_o - E)}{\hbar^2}$, we obtain

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} - \alpha^2\psi = 0$$

Its general solution is

$$\psi_{III}(x) = Ce^{\alpha x} + De^{-\alpha x}$$

When $x \to \infty$, then $e^{\alpha x} \to \infty$, so it cannot exist, therefore

$$\boxed{\psi_{III}(x) = De^{-\alpha x}}$$

(**Answer**)

(b) By applying first boundary condition at $x = L$, we obtain

$$\begin{aligned} \psi_{II}(x)\Big|_{x=L} &= \psi_{III}(x)\Big|_{x=L} \\ A\sin(kL) &= De^{\alpha L} \end{aligned} \tag{8.3}$$

And by applying second boundary condition at $x = L$, it turns out as

$$\begin{aligned} \frac{\mathrm{d}\psi_{II}(x)}{\mathrm{d}x}\Big|_{x=L} &= \frac{\mathrm{d}\psi_{III}(x)}{\mathrm{d}x}\Big|_{x=L} \\ Ak\cos(kL) &= -D\alpha e^{\alpha L} \end{aligned} \tag{8.4}$$

Upon dividing Eq. (8.4) by Eq. (8.3), we get

$$\begin{aligned} k\cot(kL) &= -\alpha \\ \sqrt{\frac{2mE}{\hbar^2}}\cot\left(\frac{\sqrt{2mE}}{\hbar}L\right) &= -\sqrt{\frac{2m(V_o - E)}{\hbar^2}} \\ \boxed{\sqrt{E}\cot\left(\frac{\sqrt{2mE}}{\hbar}L\right) = -\sqrt{V_o - E}} \end{aligned}$$

(**Answer**)