Solution Assignment 9: Quantum Field Theory

1. In the ϕ^3 scalar field theory, the Lagrangian density is given by,

$$\mathscr{L} = \frac{1}{2}(\partial^\mu \hat{\phi}(x))^2 - \frac{1}{2}m^2(\hat{\phi}(x))^2 - \frac{\eta}{3!}(\hat{\phi}(x))^3.$$

   (a) Motivate the first and second order single-particle scattering amplitudes, $\langle q|\,\hat{S}\,|p\rangle$, where $|p\rangle$ and $|q\rangle$ are input and output states. For each term in the amplitude you are required to (i) show the pairwise contraction, (ii) draw the associated Feynman diagram, (iii) identify the number of possible permutations corresponding to the same diagram and hence compute the symmetry factor$D$. [15 **marks**]

   (b) Draw at least six terms in the fourth order expansion and write down the corresponding pairwise contractions. Only fully connected Feynman diagrams are allowed. **marks**]

   (c) Consider the two particle scattering $\langle q_1 q_2|\,\hat{S}\,|p_1 p_2\rangle$. Write down the integral expansion (in momentum space) and draw the Feynman diagram corresponding to the contraction pattern.

   [10 **marks**]

   (d) Draw eight possible Feynman diagrams and the contraction patterns in the fourth order, two-particle transition amplitude. You are allowed to show both fully connected and unconnected diagrams. [10 **marks**]

Answer 1:

(a)

$$\begin{aligned}\langle q|\,\hat{S}\,|p\rangle = (2\pi)^{3/2}(2E_{\mathbf{q}})^{1/2}(2E_{\mathbf{p}})^{1/2}\Bigg[&\langle 0|\,\hat{a}_{\mathbf{q}}\hat{a}^\dagger_{\mathbf{p}}\,|0\rangle \\ &- i\int d^3z\;\frac{\eta}{3!}\tau\;\;\langle 0|\,\hat{a}_{\mathbf{q}}\;\hat{\phi}(z)\;\hat{\phi}(z)\;\hat{\phi}(z)\;\hat{a}^\dagger_{\mathbf{p}}\,|0\rangle \\ &+ \frac{1}{2!}(-i)^2\frac{\eta^3}{3!\;3!}\int d^3z\;d^3w\;\tau\;\;\langle 0|\,\hat{a}_{\mathbf{q}}\;\hat{\phi}(z)\;\hat{\phi}(z)\;\hat{\phi}(z)\hat{\phi}(z)\;\hat{\phi}(w)\;\hat{\phi}(w)\;\hat{\phi}(w)\,|0\rangle \\ &+ \dots\Bigg]\end{aligned}$$

The first order terms vanish since we have an odd number of terms in the first order expansion sandwiched between the vacuum states.

The zero'th order term yields a $\delta^{(3)}(\mathbf{p}-\mathbf{q})$ term leading to a Feynman diagram.

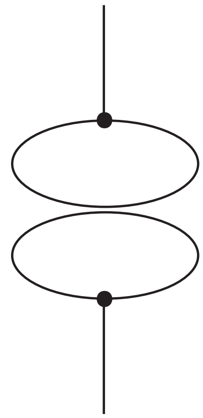
For the second order term, we need to consider the time ordered sequence.

$$\tau\,\langle 0|\,\hat{a}_{\mathbf{q}}\;\hat{\phi}(z)\;\hat{\phi}(z)\;\hat{\phi}(z)\;\hat{\phi}(w)\;\hat{\phi}(w)\;\hat{\phi}(w)\;\hat{a}_{\mathbf{p}}\;\;|0\rangle\,.$$

Below we draw the diagrams corresponding to the various choices of pairwise contraction.

i.

$$\langle 0|\ \hat{a}_{\mathbf{q}}\ \hat{\phi}(z)\ \hat{\phi}(z)\ \hat{\phi}(z)\ \hat{\phi}(w)\ \hat{\phi}(w)\ \hat{\phi}(w)\ \hat{a}^{\dagger}_{\mathbf{p}}\ |0\rangle$$

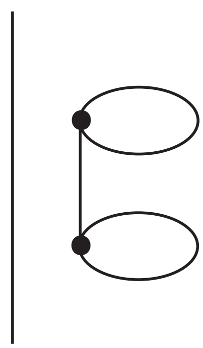


No. of possible permutations$= 3 \times 3 \times 2$.

Symmetry factor$= D = \dfrac{3 \times 3 \times 2}{2!\ 3!\ 3!} = \dfrac{3 \times 3 \times 2}{2 \times 3 \times 2 \times 3 \times 2} = \dfrac{1}{4}$.

ii.

$$\langle 0|\ \hat{a}_{\mathbf{q}}\ \hat{\phi}(z)\ \hat{\phi}(z)\ \hat{\phi}(z)\ \hat{\phi}(w)\ \hat{\phi}(w)\ \hat{\phi}(w)\ \hat{a}^{\dagger}_{\mathbf{p}}\ |0\rangle$$



No. of possible permutations$= 9$.

Symmetry factor$= D = \dfrac{9}{2!\ 3!\ 3!} = \dfrac{9}{2 \times 3 \times 2 \times 3 \times 2} = \dfrac{1}{8}$.

The 9 possible wick permutations are shown below.

(i)

$$\hat{\phi}(z)\ \hat{\phi}(z)\ \hat{\phi}(z)\ \hat{\phi}(w)\ \hat{\phi}(w)\ \hat{\phi}(w)$$

(ii)

$$\hat{\phi}(z)\,\hat{\phi}(z)\,\hat{\phi}(z)\,\hat{\phi}(w)\,\hat{\phi}(w)\,\hat{\phi}(w)$$

(iii)

$$\hat{\phi}(z)\,\hat{\phi}(z)\,\hat{\phi}(z)\,\hat{\phi}(w)\,\hat{\phi}(w)\,\hat{\phi}(w)$$

(iv)

$$\hat{\phi}(z)\,\hat{\phi}(z)\,\hat{\phi}(z)\,\hat{\phi}(w)\,\hat{\phi}(w)\,\hat{\phi}(w)$$

(v)

$$\hat{\phi}(z)\,\hat{\phi}(z)\,\hat{\phi}(z)\,\hat{\phi}(w)\,\hat{\phi}(w)\,\hat{\phi}(w)$$

(vi)

$$\hat{\phi}(z)\,\hat{\phi}(z)\,\hat{\phi}(z)\,\hat{\phi}(w)\,\hat{\phi}(w)\,\hat{\phi}(w)$$

(vii)

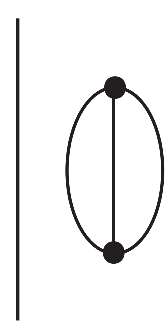
$$\hat{\phi}(z)\,\hat{\phi}(z)\,\hat{\phi}(z)\,\hat{\phi}(w)\,\hat{\phi}(w)\,\hat{\phi}(w)$$

(viii)

$$\hat{\phi}(z)\,\hat{\phi}(z)\,\hat{\phi}(z)\,\hat{\phi}(w)\,\hat{\phi}(w)\,\hat{\phi}(w)$$

(ix)

$$\hat{\phi}(z)\,\hat{\phi}(z)\,\hat{\phi}(z)\,\hat{\phi}(w)\,\hat{\phi}(w)\,\hat{\phi}(w)$$



iii.

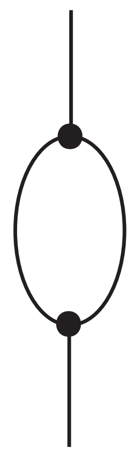
$$\hat{a}_{\mathbf{q}}\ \hat{\phi}(z)\ \hat{\phi}(z)\ \hat{\phi}(z)\ \hat{\phi}(w)\ \hat{\phi}(w)\ \hat{\phi}(w)\ \hat{a}^\dagger_{\mathbf{p}}$$

No. of possible permutations= 6.

Symmetry factor= $D = \dfrac{6}{2!\ 3!\ 3!} = \dfrac{6}{2 \times 3 \times 2 \times 3 \times 2} = \dfrac{1}{12}$.

iv.

$$\hat{a}_{\mathbf{q}}\ \hat{\phi}(z)\ \hat{\phi}(z)\ \hat{\phi}(z)\ \hat{\phi}(w)\ \hat{\phi}(w)\ \hat{\phi}(w)\ \hat{a}^\dagger_{\mathbf{p}}$$

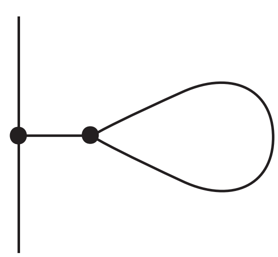


No. of possible permutations= $4 \times 3 \times 3$.

Symmetry factor= $D = \dfrac{4 \times 3 \times 3}{2!\ 3!\ 3!} = \dfrac{4 \times 3 \times 3}{2 \times 3 \times 2 \times 3 \times 2} = \dfrac{1}{2}$.

v.

$$\hat{a}_{\mathbf{q}}\ \hat{\phi}(z)\ \hat{\phi}(z)\ \hat{\phi}(z)\ \hat{\phi}(w)\ \hat{\phi}(w)\ \hat{\phi}(w)\ \hat{a}^\dagger_{\mathbf{p}}$$

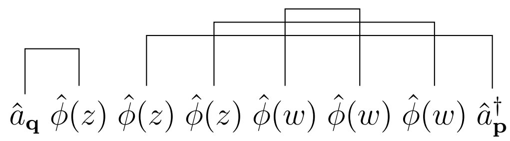


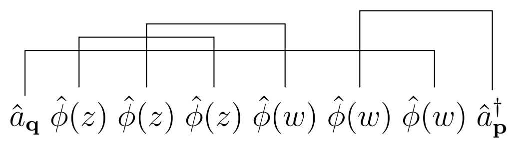
No. of possible permutations= $2 \times 3 \times 3 \times 2$.

Symmetry factor= $D = \dfrac{2 \times 3 \times 3 \times 2}{2!\ 3!\ 3!} = \dfrac{2 \times 3 \times 3 \times 2}{2 \times 3 \times 2 \times 3 \times 2} = \dfrac{1}{2}$.

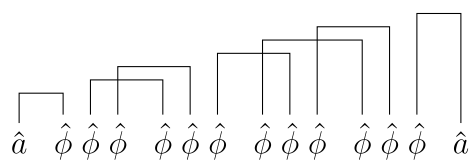
Why are these 36 permutations?

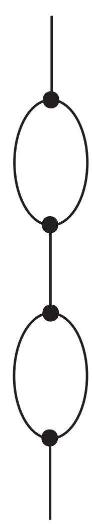
$\hat{a}_{\mathbf{p}}$ can connect with one of 3 $\hat{\phi}(z)$'s giving 3 possibilities. The $\hat{a}_{\mathbf{q}}$ can connect with either of the two remaining $\hat{\phi}(z)$'s giving 2 permutations. The remaining unconnected $\hat{\phi}(z)$ can bridge over to either of the three $\hat{\phi}(z)$'s yielding 3 possibilities. Finally, we can choose to swap the connections of $\hat{a}_{\mathbf{q}}$ and $\hat{a}_{\mathbf{p}}$ yielding two more possibilities. Overall, the number of permutations comes out as $2 \times 3 \times 3 \times 2 = 36$. Some of these are shown below.



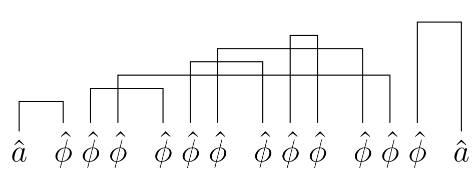


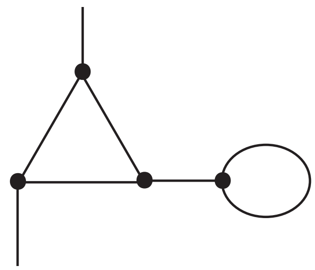
(b) (i)





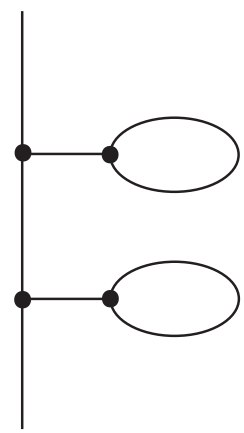
(ii)





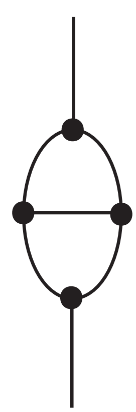
(iii)

$$\hat{a} \quad \hat{\phi}\,\hat{\phi}\,\hat{\phi} \quad \hat{\phi}\,\hat{\phi}\,\hat{\phi} \quad \hat{\phi}\,\hat{\phi}\,\hat{\phi} \quad \hat{\phi}\,\hat{\phi}\,\hat{\phi} \quad \hat{a}$$



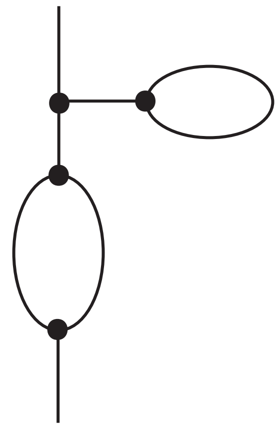
(iv)

$$\hat{a} \quad \hat{\phi}\,\hat{\phi}\,\hat{\phi} \quad \hat{\phi}\,\hat{\phi}\,\hat{\phi} \quad \hat{\phi}\,\hat{\phi}\,\hat{\phi} \quad \hat{\phi}\,\hat{\phi}\,\hat{\phi} \quad \hat{a}$$



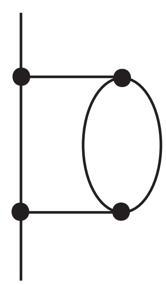
(v)

$$\hat{a} \quad \hat{\phi}\,\hat{\phi}\,\hat{\phi} \quad \hat{\phi}\,\hat{\phi}\,\hat{\phi} \quad \hat{\phi}\,\hat{\phi}\,\hat{\phi} \quad \hat{\phi}\,\hat{\phi}\,\hat{\phi} \quad \hat{a}$$



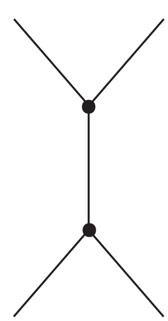
(vi)

$$\hat{a} \quad \hat{\phi}\,\hat{\phi}\,\hat{\phi} \quad \hat{\phi}\,\hat{\phi}\,\hat{\phi} \quad \hat{\phi}\,\hat{\phi}\,\hat{\phi} \quad \hat{\phi}\,\hat{\phi}\,\hat{\phi} \quad \hat{a}$$



vi.

$$\langle 0|\hat{a}_{\mathbf{q_1}}\ \hat{a}_{\mathbf{q_2}}\ \hat{\phi}(z)\ \hat{\phi}(z)\ \hat{\phi}(z)\ \hat{\phi}(w)\ \hat{\phi}(w)\ \hat{\phi}(w)\ \hat{a}^\dagger_{\mathbf{p_2}}\ \hat{a}^\dagger_{\mathbf{p_1}}|0\rangle$$



$$\tau \langle 0|\, \hat{a}_{\mathbf{q}_1}\hat{\phi}(z)\,|0\rangle\ \ \tau \langle 0|\, \hat{a}_{\mathbf{q}_2}\hat{\phi}(z)\,|0\rangle\ \ \tau \langle 0|\, \hat{\phi}(z)\hat{\phi}(w)\,|0\rangle\ \ \tau \langle 0|\, \hat{\phi}(w)\hat{a}^\dagger_{\mathbf{p}_2}\,|0\rangle\ \ \tau \langle 0|\, \hat{\phi}(w)\hat{a}^\dagger_{\mathbf{p}_1}\,|0\rangle\,.$$

The second-order, two particle interaction amplitude can be computed as follows.

$$
\begin{aligned}
\tau \langle 0|\, \hat{a}_{\mathbf{q_1}} \hat{\phi}(z)\, |0\rangle &= \int \frac{d^3p}{(2\pi)^{3/2}(2E_{\mathbf{p}})^{1/2}} \langle 0|\, \hat{a}_{\mathbf{q_1}} \hat{a}^\dagger_{\mathbf{p}}\, |0\rangle\, e^{+ip\cdot z} \\
&= \frac{e^{+iq_1\cdot z}}{(2\pi)^{3/2}(2E_{\mathbf{q_1}})^{1/2}} \delta^{(3)}(\mathbf{p} - \mathbf{q}_1) \\
&= \frac{e^{+iq_1\cdot z}}{(2\pi)^{3/2}(2E_{\mathbf{q_1}})^{1/2}}
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\tau \langle 0|\, \hat{a}_{\mathbf{q_2}} \hat{\phi}(z)\, |0\rangle &= \frac{e^{+iq_2\cdot z}}{(2\pi)^{3/2}(2E_{\mathbf{q_2}})^{1/2}} \\
\tau \langle 0|\, \hat{\phi}(z)\hat{\phi}(w)\, |0\rangle &= \int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(z-w)} \\
\tau \langle 0|\, \hat{\phi}(w)\hat{a}^\dagger_{\mathbf{p_2}}\, |0\rangle &= \frac{e^{-ip_2\cdot w}}{(2\pi)^{3/2}(2E_{\mathbf{p_2}})^{1/2}} \\
\tau \langle 0|\, \hat{\phi}(w)\hat{a}^\dagger_{\mathbf{p_1}}\, |0\rangle &= \frac{e^{-ip_1\cdot w}}{(2\pi)^{3/2}(2E_{\mathbf{p_1}})^{1/2}}
\end{aligned}
$$

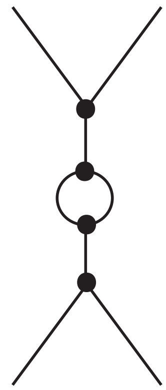
The transition amplitude will be proportional to (ignoring the symmetry factors),

$$
\begin{aligned}
\int d^4z\; d^4w\; e^{i(q_1+q_2)\cdot z} &\int \frac{d^4p}{(2\pi)^4}\; e^{-ip\cdot(z-w)}\; e^{-i(p_1+p_2)\cdot w} \\
&= \int d^4z\; d^4w\; \frac{d^4p}{(2\pi)^4}\; e^{i(q_1+q_2-p)\cdot z}\; e^{-i(p_1+p_2-p)\cdot w} \\
&= (2\pi)^4(2\pi)^4 \int \frac{d^4p}{(2\pi)^4} \delta^{(4)}(p - q_1 - q_2)\; \delta^{(4)}(p - p_1 - p_2)
\end{aligned}
$$

This perfectly makes sense as it predicates that the incoming momentum p_1+p_2 must match the outgoing momentum $q_1 + q_2$. Furthermore, momentum is also conserved at each vertex. Furthermore, p is an unconstrained momentum in this diagram.

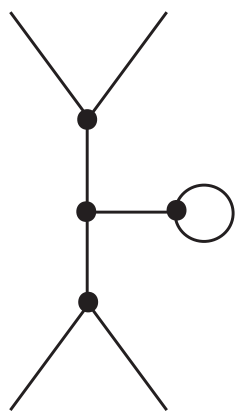
(d) (i)

$$
\hat{a}_{\mathbf{q_1}}\ \hat{a}_{\mathbf{q_2}}\ \hat{\phi}\ \hat{\phi}\ \hat{\phi}\ \hat{\phi}\ \hat{\phi}\ \hat{\phi}\ \hat{\phi}\ \hat{\phi}\ \hat{\phi}\ \hat{\phi}\ \hat{\phi}\ \hat{\phi}\ \hat{a}^\dagger_{\mathbf{p_2}}\ \hat{a}^\dagger_{\mathbf{p_1}}
$$



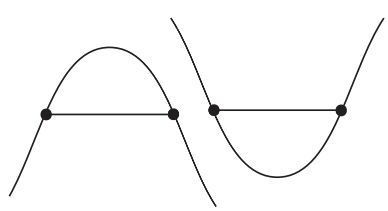
(ii)

$$\hat{a}_{\mathbf{q_1}}\ \hat{a}_{\mathbf{q_2}}\quad \hat{\phi}\,\hat{\phi}\,\hat{\phi}\quad \hat{\phi}\,\hat{\phi}\,\hat{\phi}\quad \hat{\phi}\,\hat{\phi}\,\hat{\phi}\quad \hat{\phi}\,\hat{\phi}\,\hat{\phi}\quad \hat{a}^{\dagger}_{\mathbf{p_2}}\ \hat{a}^{\dagger}_{\mathbf{p_1}}$$



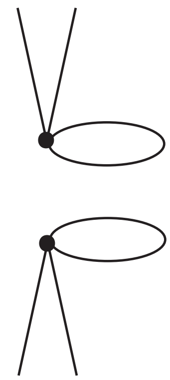
(iii)

$$\hat{a}_{\mathbf{q_1}}\ \hat{a}_{\mathbf{q_2}}\quad \hat{\phi}\,\hat{\phi}\,\hat{\phi}\quad \hat{\phi}\,\hat{\phi}\,\hat{\phi}\quad \hat{\phi}\,\hat{\phi}\,\hat{\phi}\quad \hat{\phi}\,\hat{\phi}\,\hat{\phi}\quad \hat{a}^{\dagger}_{\mathbf{p_2}}\ \hat{a}^{\dagger}_{\mathbf{p_1}}$$



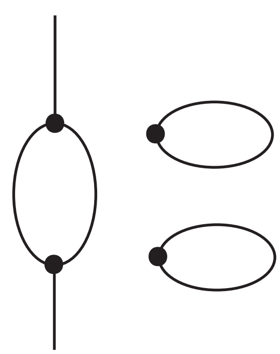
(iv)

$$\hat{a}_{\mathbf{q_1}}\,\hat{a}_{\mathbf{q_2}}\;\hat{\phi}\,\hat{\phi}\,\hat{\phi}\;\hat{\phi}\,\hat{\phi}\,\hat{\phi}\;\hat{\phi}\,\hat{\phi}\,\hat{\phi}\;\hat{\phi}\,\hat{\phi}\,\hat{\phi}\;\hat{a}^\dagger_{\mathbf{p_2}}\,\hat{a}^\dagger_{\mathbf{p_1}}$$



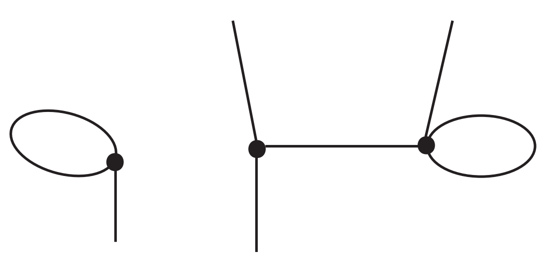
(v)

$$\hat{a}_{\mathbf{q_1}}\,\hat{a}_{\mathbf{q_2}}\;\hat{\phi}\,\hat{\phi}\,\hat{\phi}\;\hat{\phi}\,\hat{\phi}\,\hat{\phi}\;\hat{\phi}\,\hat{\phi}\,\hat{\phi}\;\hat{\phi}\,\hat{\phi}\,\hat{\phi}\;\hat{a}^\dagger_{\mathbf{p_2}}\,\hat{a}^\dagger_{\mathbf{p_1}}$$



(vi)

$$\hat{a}_{\mathbf{q_1}}\,\hat{a}_{\mathbf{q_2}}\;\hat{\phi}\,\hat{\phi}\,\hat{\phi}\;\hat{\phi}\,\hat{\phi}\,\hat{\phi}\;\hat{\phi}\,\hat{\phi}\,\hat{\phi}\;\hat{\phi}\,\hat{\phi}\,\hat{\phi}\;\hat{a}^\dagger_{\mathbf{p_2}}\,\hat{a}^\dagger_{\mathbf{p_1}}$$



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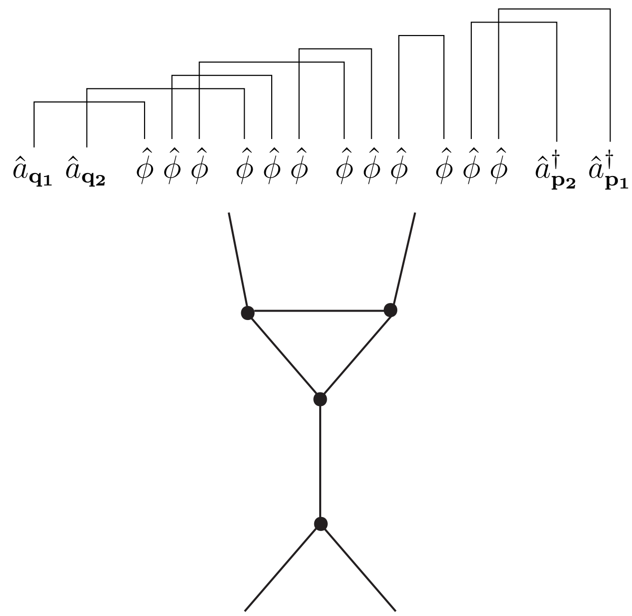
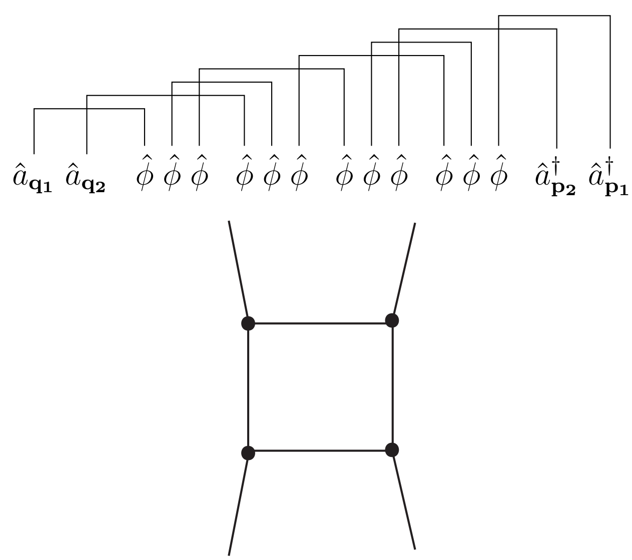


(viii)



(b) The ABA theory involves interaction between two types of scalar fields, labeled A and B. The Lagrangian density is,

$$\mathscr{L} \;=\; \frac{1}{2}(\partial^{\mu}\hat{\phi}_A(x))^2 - \frac{1}{2}m_A^2(\hat{\phi}_A(x))^2 + \frac{1}{2}(\partial^{\mu}\hat{\phi}_B(x))^2 - \frac{1}{2}m_B^2(\hat{\phi}_B(x))^2 - \frac{g}{2}\hat{\phi}_A(x)\hat{\phi}_B(x)\hat{\phi}_A(x).$$

i. Draw Feynman diagram for the following second order contractions. [5 **marks**]

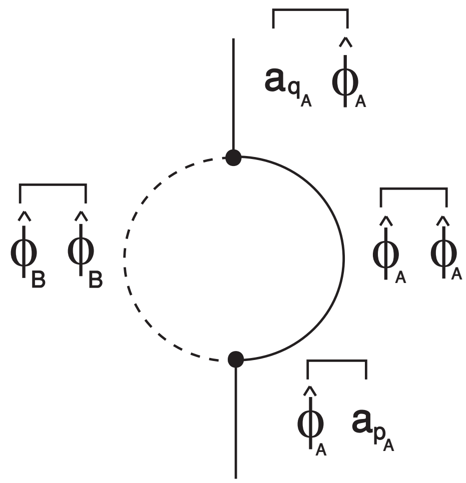
ii. Write down the momentum-space integrals for the propagation corresponding to these Feynman diagrams. [5 **marks**]

Answer 2:

i. Here are the various contractions and their corresponding Feynman diagrams.

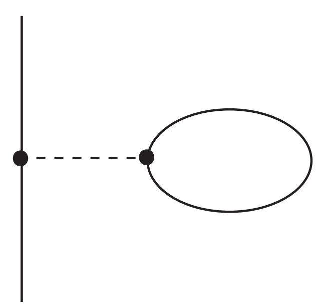
A.

$$\hat{a}_{qA}\,\hat{\phi}_A(z)\,\hat{\phi}_B(z)\,\hat{\phi}_A(z)\,\hat{\phi}_A(w)\,\hat{\phi}_B(w)\,\hat{\phi}_A(w)\,\hat{a}_{pA}$$



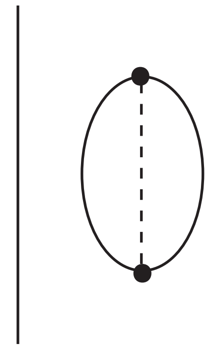
B.

$$\hat{a}_{qA}\,\hat{\phi}_A(z)\,\hat{\phi}_B(z)\,\hat{\phi}_A(z)\,\hat{\phi}_A(w)\,\hat{\phi}_B(w)\,\hat{\phi}_A(w)\,\hat{a}_{pA}$$



C.

$$\hat{a}_{qA}\,\hat{\phi}_A(z)\,\hat{\phi}_B(z)\,\hat{\phi}_A(z)\,\hat{\phi}_A(w)\,\hat{\phi}_B(w)\,\hat{\phi}_A(w)\,\hat{a}_{pA}$$



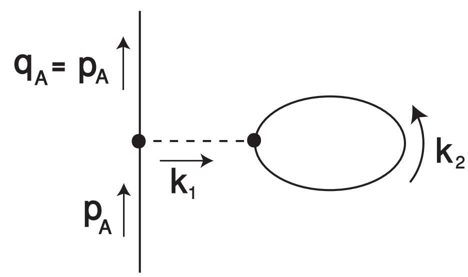
ii. A.

$$(-ig)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m_A^2 + i\delta} \frac{i}{(p_A - k)^2 - m_B^2 + i0^+} \delta^4(q_A - p_A)$$

B.

$$(-ig)^2 \int \frac{d^4k_1}{(2\pi)^4} \frac{i}{k_1^2 - m_B^2 + i0^+} \frac{d^4k_2}{(2\pi)^4} \frac{i}{k_2^2 - m_A^2 + i0^+} \delta^4(k_1)$$

k_1 must be zero by momentum conservation.



C.

$$\int \frac{d^4k_1}{(2\pi)^4} \frac{i}{k_1^2 - m_A^2 + i0^+} \frac{d^4k_2}{(2\pi)^4} \frac{i}{k_2^2 - m_A^2 + i0^+} \frac{i}{(-k_1^2 - k_2^2) - m_B^2 + i0^+}$$