Solution Assignment 3: Modern Physics

1. **Answer 1:**

   The qubit is given by,

$$\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

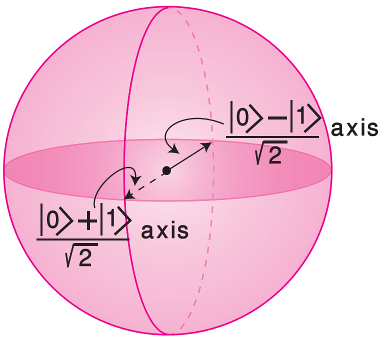
   Compare it with general form of qubit,

$$\cos\left(\frac{\theta}{2}\right)|\,0\rangle \;+\; e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\,1\rangle$$

   It can be seen that,

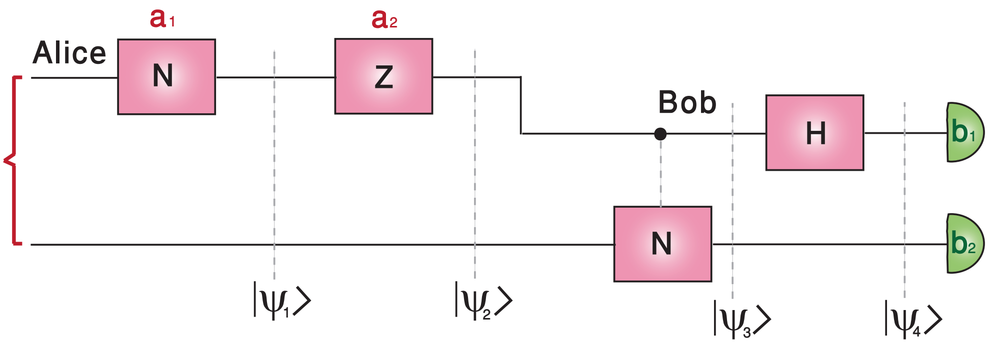
$$\Rightarrow \quad \cos\left(\frac{\theta}{2}\right) = \frac{1}{\sqrt{2}}$$
$$\frac{\theta}{2} = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}\ .$$

   So $\left(\theta = \frac{\pi}{2}\right)$ and the state lies on the equatorial plane. We now need to determine ϕ and it is clear by comparison that $\phi = \pi$.



   It is clear that the state could be transformed by a rotation of 180° around the $|0\rangle$ axis, or 180° rotation around the $\dfrac{|0\rangle + i\,|1\rangle}{\sqrt{2}}$ axis into $\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ or 180° around the $\dfrac{|0\rangle - i\,|1\rangle}{\sqrt{2}}$ axis, etc.

2. **Answer 2:** In order to answer this question, consider the accompanying figure.



(a) **Case 1:** $(a_1, a_2) = (0, 0)$ As $a_1 = 0$ and $a_2 = 0$, Alice's qubit will not change and,

$$
\begin{aligned}
|\psi_2\rangle &= \frac{|\,0\rangle|\,1\rangle + |\,1\rangle|\,0\rangle}{\sqrt{2}} \\
|\psi_3\rangle &= \frac{|\,0\rangle|\,1\rangle + |\,1\rangle|\,1\rangle}{\sqrt{2}} \\
|\psi_4\rangle &= \frac{\left(\frac{|\,0\rangle + |\,1\rangle}{\sqrt{2}}\right)|1\rangle + \left(\frac{|\,0\rangle - |\,1\rangle}{\sqrt{2}}\right)|1\rangle}{\sqrt{2}} \\
&= \frac{1}{2}\Big[|\,0\rangle|\,1\rangle + |\,1\rangle|1\rangle + |\,0\rangle|1\rangle - |\,1\rangle|1\rangle\Big] \\
&= |\,0\rangle|1\rangle \\
\Rightarrow \qquad b_1 &= 0, \qquad b_2 = 1.
\end{aligned}
$$

Case 2: $(a_1, a_2) = (0, 1)$

$$
\begin{aligned}
|\psi_2\rangle &= \frac{|\,0\rangle|\,1\rangle - |\,1\rangle|\,0\rangle}{\sqrt{2}} \\
|\psi_3\rangle &= \frac{|\,0\rangle|\,1\rangle - |\,1\rangle|\,1\rangle}{\sqrt{2}} \\
|\psi_3\rangle &= \frac{\left(\frac{|\,0\rangle + |\,1\rangle}{\sqrt{2}}\right)|1\rangle - \left(\frac{|\,0\rangle - |\,1\rangle}{\sqrt{2}}\right)|1\rangle}{\sqrt{2}} \\
&= \frac{1}{2}\Big[|\,0\rangle|\,1\rangle + |\,1\rangle|1\rangle - |\,0\rangle|1\rangle + |\,1\rangle|1\rangle\Big] \\
&= |\,1\rangle|1\rangle \\
\Rightarrow \qquad b_1 &= 1, \qquad b_2 = 1.
\end{aligned}
$$

Case 3: $(a_1, a_2) = (1, 0)$

$$
\begin{aligned}
|\psi_1\rangle &= \frac{|\,1\rangle|\,1\rangle + |\,0\rangle|\,0\rangle}{\sqrt{2}} \\
|\psi_2\rangle &= \frac{|\,1\rangle|\,1\rangle + |\,0\rangle|\,0\rangle}{\sqrt{2}} \\
|\psi_3\rangle &= \frac{|\,1\rangle|\,0\rangle + |\,0\rangle|\,0\rangle}{\sqrt{2}} \\
|\psi_4\rangle &= \frac{\left(\frac{|\,0\rangle - |\,1\rangle}{\sqrt{2}}\right)|0\rangle - \left(\frac{|\,0\rangle + |\,1\rangle}{\sqrt{2}}\right)|0\rangle}{\sqrt{2}} \\
&= \frac{1}{2}\Big[|\,0\rangle|\,0\rangle - |\,1\rangle|0\rangle + |\,0\rangle|0\rangle + |\,1\rangle|0\rangle\Big] \\
&= |\,0\rangle|0\rangle \\
\Rightarrow \qquad b_1 &= 0, \quad b_2 = 0.
\end{aligned}
$$

Case 4: $(a_1, a_2) = (1, 1)$

$$
\begin{aligned}
|\psi_1\rangle &= \frac{|\,1\rangle|\,1\rangle + |\,0\rangle|\,0\rangle}{\sqrt{2}} \\
|\psi_2\rangle &= \frac{-|\,1\rangle|\,1\rangle + |\,0\rangle|\,0\rangle}{\sqrt{2}} \\
|\psi_3\rangle &= \frac{-|\,1\rangle|\,0\rangle + |\,0\rangle|\,0\rangle}{\sqrt{2}} \\
|\psi_4\rangle &= \frac{-\left(\frac{|\,0\rangle - |\,1\rangle}{\sqrt{2}}\right)|0\rangle + \left(\frac{|\,0\rangle + |\,1\rangle}{\sqrt{2}}\right)|0\rangle}{\sqrt{2}} \\
&= \frac{1}{2}\Big[-|\,0\rangle|\,0\rangle + |\,1\rangle|0\rangle + |\,0\rangle|0\rangle + |\,1\rangle|0\rangle\Big] \\
&= |\,1\rangle|0\rangle \\
\Rightarrow \qquad b_1 &= 1, \quad b_2 = 0.
\end{aligned}
$$

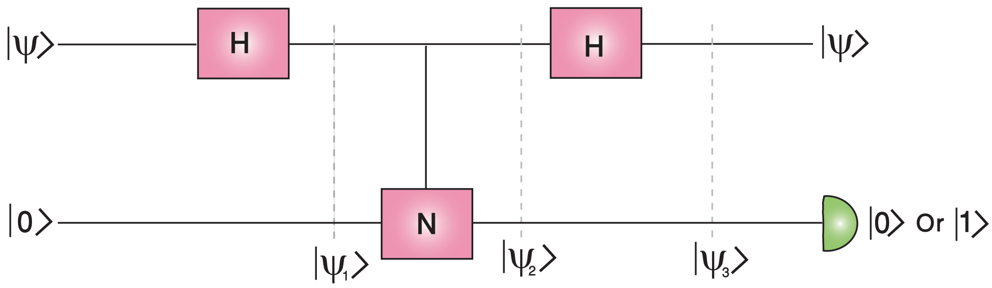
Hence the outcome received by Bob is is reflected of the message (a_1, a_2) generated by Alice. The formation (a_1, a_2) is successfully transmitted to Bob as (b_1, b_2) according to the following one-to-one mapping.

a_1 a_2	b_1 b_2
0 0	0 1
0 1	1 1
1 0	0 0
1 1	1 0

(b) See previous note.

(c) Half of the time he can tap on the information of only one bit but he has no means to confirm the other bit.

3. Consider the diagram below which shows a circuit that can distinguish between $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$.



Scenario I:

$$
\begin{aligned}
|\psi\rangle_{in} &= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|0\rangle \\
|\psi\rangle_1 &= \frac{1}{\sqrt{2}}\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|0\rangle + \frac{1}{\sqrt{2}}\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)|0\rangle = |0\rangle\,|0\rangle \\
|\psi\rangle_2 &= |0\rangle\,|0\rangle \\
|\psi\rangle_3 &= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|0\rangle \\
|\psi\rangle_3 &= \frac{|0\rangle\,|0\rangle + |1\rangle\,|0\rangle}{\sqrt{2}}
\end{aligned}
$$

Scenario II:

$$
\begin{aligned}
|\psi\rangle_{in} &= \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)|0\rangle \\
|\psi\rangle_1 &= \frac{1}{\sqrt{2}}\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|0\rangle - \frac{1}{\sqrt{2}}\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)|0\rangle = |1\rangle\,|0\rangle \\
|\psi\rangle_2 &= |1\rangle\,|1\rangle \\
|\psi\rangle_3 &= \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right)|1\rangle \\
|\psi\rangle_3 &= \frac{|0\rangle\,|1\rangle - |1\rangle\,|1\rangle}{\sqrt{2}}
\end{aligned}
$$

The second qubit is now detected as $|1\rangle$, which is orthogonal to scenario I. Hence, this circuit clearly discriminates between $\frac{|0\rangle + |1\rangle}{\sqrt{2}}$ and $\frac{|0\rangle - |1\rangle}{\sqrt{2}}$.

4. (a)

$$\begin{aligned}
|z\rangle &= \frac{1}{\sqrt{2}}\Big[|x\rangle + |-x\rangle\Big] \\
\text{Now} \quad \langle z|-z\rangle &= 0 \\
\text{Suppose} \quad |-z\rangle &= c_1|x\rangle + c_2|-x\rangle \\
\langle z|-z\rangle &= \Big[\langle x|\frac{1}{\sqrt{2}} + \langle -x|\frac{1}{\sqrt{2}}\Big]\Big[c_1|x\rangle + c_2|-x\rangle\Big] \\
&= \langle x|x\rangle\frac{c_1}{\sqrt{2}} + \langle x|-x\rangle\frac{c_2}{\sqrt{2}} + \langle -x|x\rangle\frac{c_1}{\sqrt{2}} + \langle -x|-x\rangle\frac{c_2}{\sqrt{2}}
\end{aligned}$$

We are given that $\langle x|-x\rangle = 0$, $\langle x|x\rangle = 1$ and $\langle -x|-x\rangle = 1$, so

$$\begin{aligned}
\langle z|-z\rangle &= \frac{c_1}{\sqrt{2}} + \frac{c_2}{\sqrt{2}} = 0 \\
\Rightarrow \quad c_1 + c_2 &= 0 \\
\Rightarrow \quad c_1 &= -c_2.
\end{aligned}$$

Also,

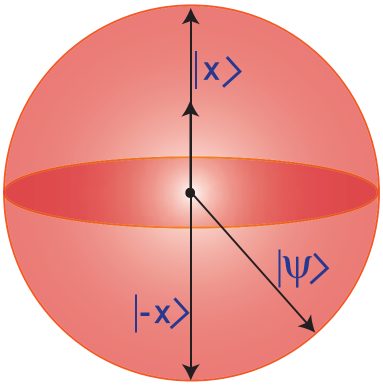
$$\begin{aligned}
\left|c_1\right|^2 + \left|c_2\right|^2 &= 1 \\
\left|-c_2\right|^2 + \left|c_2\right|^2 &= 1 \\
c_2^2 + c_2^2 &= 1 \\
2c_2^2 &= 1 \\
c_2^2 &= \frac{1}{2} \\
c_2 &= \frac{1}{\sqrt{2}} \\
\Rightarrow \quad |-z\rangle &= \frac{-1}{\sqrt{2}}|x\rangle + \frac{1}{\sqrt{2}}|-x\rangle \\
&= \frac{1}{\sqrt{2}}\Big[|x\rangle - |-x\rangle\Big].
\end{aligned}$$

(b)

$$|\psi\rangle = \frac{2}{\sqrt{13}}|x\rangle + \frac{3i}{\sqrt{13}}|-x\rangle$$

(c)

$$\begin{aligned}
\cos\frac{\theta}{2} &= \left(\frac{2}{\sqrt{13}}\right) \\
\Rightarrow \quad \theta &= 2\cos^{-1}\left(\frac{2}{\sqrt{13}}\right) = \pm 112.62^\circ \\
\theta &= 2\sin^{-1}\left(\frac{3}{\sqrt{13}}\right) = +112.62^\circ \\
e^{i\phi} &= i \quad \Rightarrow \quad \sin\phi = 1 \\
\phi &= \sin^{-1}(1) = \frac{\pi}{2}.
\end{aligned}$$



(d) As the state is already along x direction, the probabilities would be simply

$$\begin{aligned}
\mathrm{P}(|x\rangle) &= \left|\left(\frac{2}{\sqrt{13}}\right)\right|^2 = \frac{4}{13} \\
\mathrm{P}(|-x\rangle) &= \left|\left(i\frac{3}{\sqrt{13}}\right)\right|^2 = \frac{9}{13}
\end{aligned}$$

(e) It is given that

$$|\psi\rangle = \frac{2}{\sqrt{13}}|x\rangle + i\frac{3}{\sqrt{13}}|-x\rangle$$

Now, if it enters Stern–Gerlach apparatus along the z-axis, it is convenient to express $|z\rangle$ and $|-z\rangle$ as superposition of $|x\rangle$ and $|-x\rangle$ as given below.

$$\begin{aligned}
|z\rangle &= \frac{1}{\sqrt{2}}\Big(|x\rangle + |-x\rangle\Big) \\
|-z\rangle &= \frac{1}{\sqrt{2}}\Big(|x\rangle - |-x\rangle\Big)
\end{aligned}$$

So

$$|z\rangle + |-z\rangle = \frac{1}{\sqrt{2}}\Big(2|x\rangle\Big)$$

It gives

$$|x\rangle \;=\; \frac{1}{\sqrt{2}}\Big(|z\rangle + |{-z}\rangle\Big)$$

and

$$|-x\rangle \;=\; \frac{1}{\sqrt{2}}\Big(|z\rangle - |{-z}\rangle\Big)$$

So

$$\begin{aligned}
|\psi_{\text{out}}\rangle \;&=\; \frac{2}{\sqrt{13}}\left(\frac{1}{\sqrt{2}}\Big(|z\rangle + |-z\rangle\Big)\right) + i\frac{3}{\sqrt{13}}\left(\frac{1}{\sqrt{2}}\Big(|z\rangle - |-z\rangle\Big)\right) \\
&=\; \left(\frac{2}{\sqrt{13}}\left(\frac{1}{\sqrt{2}}\right) + i\frac{3}{\sqrt{13}}\frac{1}{\sqrt{2}}\right)|z\rangle \\
&=\; \left(\frac{2}{\sqrt{13}}\left(\frac{1}{\sqrt{2}}\right) - i\frac{3}{\sqrt{13}}\frac{1}{\sqrt{2}}\right)|-z\rangle \\
|\psi_{\text{out}}\rangle \;&=\; \frac{2+3i}{\sqrt{2}\sqrt{13}}|z\rangle + \frac{2-3i}{\sqrt{2}\sqrt{13}}|-z\rangle
\end{aligned}$$

Thus

$$\begin{aligned}
\text{Prob}(|z\rangle) \;&=\; \left|\frac{2+3i}{\sqrt{2}\sqrt{13}}\right|^2 = \left(\frac{2-3i}{\sqrt{2}\sqrt{13}}\right)\left(\frac{2+3i}{\sqrt{2}\sqrt{13}}\right) \\
&=\; \frac{2^2-(3i)^2}{26} \\
&=\; \frac{13}{26} \\
\text{Prob}(|z\rangle) \;&=\; \frac{1}{2}
\end{aligned}$$

Similarly

$$\text{Prob}(|-z\rangle) \;=\; \frac{1}{2}$$