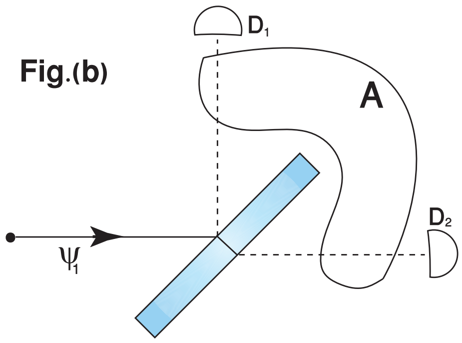
Solution Tutorial 3: Modern Physics

1. **Answer 1:**

part (i):

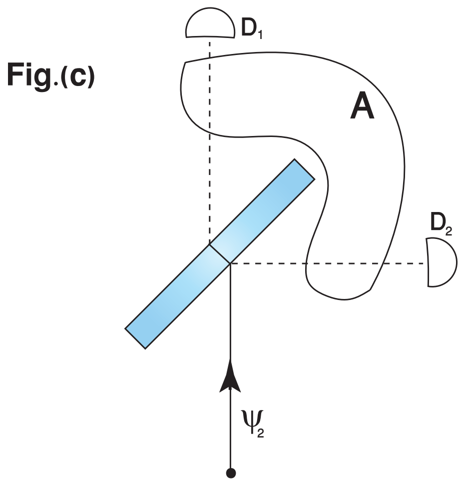


Since we have a 50:50 beamsplitter, there is an equal chance of any one of the detectors clicking, but only one at a time. This can also be observed by noting that the wavefunction in the region A is $\frac{1}{\sqrt{2}}(\psi_1 + \psi_2)$. Hence the probability of D_1 clicking is $(\frac{1}{\sqrt{2}})^2 = \frac{1}{2}$ and of D_2 clicking is also $(\frac{1}{\sqrt{2}})^2 = \frac{1}{2}$.

part (ii):

As many photons come in, both the detectors will click with equal probabilities but never simultaneously. Detection collapses the wavefunction which then appears as a "particle". The particle—called the photon—cannot be split. So never two detectors will click at the same time.

part (iii):



The answer to (iii) is identical to (i) and (ii). In the region A, the wavefunction is $\frac{1}{\sqrt{2}}(\psi_1 - \psi_2)$. The minus sign before ψ_2 does not make any physical difference here because the probability "washes" away the minus sign, i.e. $(-\frac{1}{\sqrt{2}})^2 = (\frac{1}{\sqrt{2}})^2 = \frac{1}{2}$.

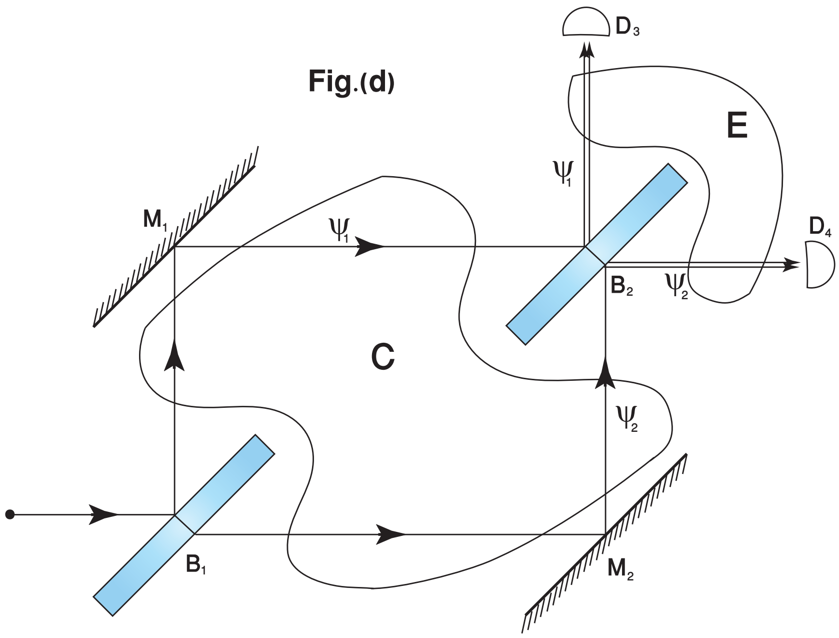
part (iv):

The superposition state is such that the coefficients of ψ_1 and ψ_2, when modulus squared, yield 0.7 and 0.3. Hence one possible superposition field for a 70:30 beam-splitter is $\sqrt{0.7}\psi_1 + \sqrt{0.3}\psi_2$. Other possibilities are,

$$-\sqrt{0.7}\psi_1 - \sqrt{0.3}\psi_2$$
$$i\sqrt{0.7}\psi_1 + i\sqrt{0.3}\psi_2, \quad \text{and}$$
$$-i\sqrt{0.7}\psi_1 - i\sqrt{0.3}\psi_2.$$

You will immediately notice that in each case, the probabilities come out as correct, e.g., $(\sqrt{0.7})^2 = |-\sqrt{0.7}|^2 = |i\sqrt{0.7}|^2 = |-i\sqrt{0.7}|^2 = 0.7$

part (v):



When the incident photon emerges from B_1, The wavefunction in the region C is,

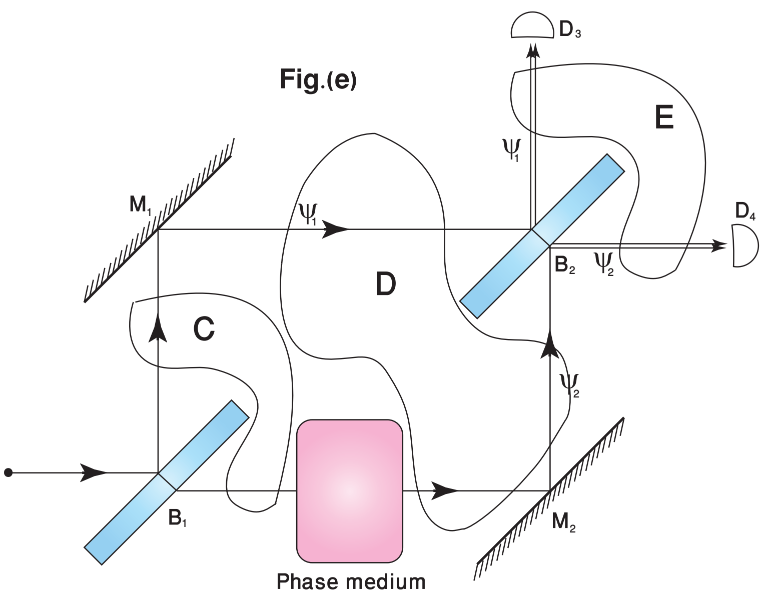
$$\psi_1 \xrightarrow{\mathrm{B_1}} \frac{1}{\sqrt{2}}(\psi_1 + \psi_2).$$

Now the wavefunction in the region E (before collapse),

$$\begin{aligned}\frac{1}{\sqrt{2}}(\psi_1+\psi_2) \ \xrightarrow{\mathrm{B}_2}\ & \frac{1}{\sqrt{2}}\left[\frac{\psi_1+\psi_2}{\sqrt{2}}+\frac{\psi_1-\psi_2}{\sqrt{2}}\right] \\ =\ & \frac{1}{2}(\psi_1+\psi_2+\psi_1-\psi_2) \\ =\ & \frac{1}{2}(2\psi_1) \\ =\ & \psi_1.\end{aligned}$$

Since the emergent wavefunction is ψ_1 close to the region of detectors, only D_3 will click with 100% probability. This strange outcome is a result of destructive interference between the component fields resulting in complete annihilation of the field ψ_2 in the region close to the detectors. This interference is caused by the second beamsplitter while the first beamsplitter creates a superposition.

part (vi):



To answer this question, we can identify three regions C, D and E inside the interferometer. In region C the field is $\frac{\psi_1+\psi_2}{\sqrt{2}}$. In region D the ψ_2 component of the field takes up a phase factor $e^{i\delta}$, the resulting field is,

$$\frac{1}{\sqrt{2}}(\psi_1+e^{i\delta}\psi_2).$$

In E, after the second beamsplitter the field is,

$$\frac{1}{\sqrt{2}}\left(\frac{\psi_1+\psi_2}{\sqrt{2}}+e^{i\delta}\frac{\psi_1-\psi_2}{\sqrt{2}}\right).$$

The action of the beamsplitters is unchanged, only that a phase factor has been added. Simplifying the above,

$$\begin{aligned}
\frac{1}{2}\Big(\psi_1(1+e^{i\delta})+\psi_2(1-e^{i\delta})\Big) &= \frac{e^{i\delta/2}}{2}\Big(\psi_1(e^{-i\delta/2}+e^{i\delta/2})+\psi_2(e^{-i\delta/2}-e^{i\delta/2})\Big)\\
&= e^{i\delta/2}\Big(\psi_1(\frac{e^{-i\delta/2}+e^{i\delta/2}}{2})+\psi_2(\frac{e^{-i\delta/2}-e^{i\delta/2}}{2})\Big)\\
&= e^{i\delta/2}\Big(\psi_1(\frac{e^{-i\delta/2}+e^{i\delta/2}}{2})-\psi_2(\frac{e^{i\delta/2}-e^{-i\delta/2}}{2})\Big)\\
&= e^{i\delta/2}\big(\cos(\delta/2)\psi_1 - i\sin(\delta/2)\psi_2\big).
\end{aligned}$$

The coefficient of ψ_1 is $e^{i\delta/2}\cos(\delta/2)$ and coefficient of ψ_2 is $-ie^{i\delta/2}\sin(\delta/2)$.

$$\begin{aligned}
\text{prob } (D_3 \text{ clicking}) &= \left|e^{i\delta/2}\cos\delta/2\right|^2\\
&= e^{-i\delta/2}\cos\delta/2\cdot e^{i\delta/2}\cos\delta/2\\
&= \cos^2\delta/2\\
\text{prob } (D_4 \text{ clicking}) &= \left|-ie^{i\delta/2}\sin\delta/2\right|^2\\
&= ie^{-i\delta/2}\sin\delta/2\cdot -ie^{i\delta/2}\sin\delta/2\\
&= \sin^2\delta/2.
\end{aligned}$$

Notice that Prob(D_3 clicking)+Prob(D_4 clicking)$= 1$ as expected.

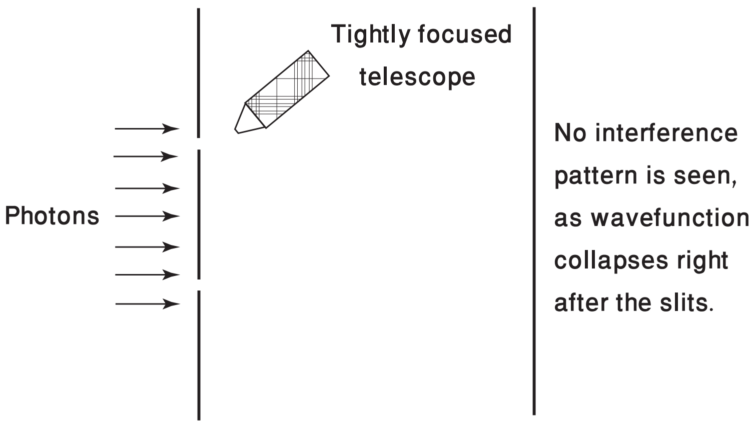
Yes we can make the detectors click with equal probability by adjusting δ to $\delta = \pi/2$, then Prob(D_3 clicking)=Prob(D_4 clicking)$= \cos^2(\pi/4) = \sin^2(\pi/4) = 1/2$

part (vii):

D_2 provides which path information after B_1, collapsing the wavefunctions. Hence D_2 should click 50% of the times. When D_2 does not click, the photon takes the upper arm and is converted into a superposition $\frac{\psi_1+\psi_2}{\sqrt{2}}$ get again. But D_3 and D_4 collapse this wavefunction again, with equal probabilities of each of these detectors clicking. Hence

$$\begin{aligned}
\text{Prob}(D_2 \text{ click}) &= 0.5\\
\text{Prob}(D_3 \text{ click}) &= 0.5\times 0.5 = 0.25\\
\text{Prob}(D_4 \text{ click}) &= 0.25.
\end{aligned}$$

part (viii):

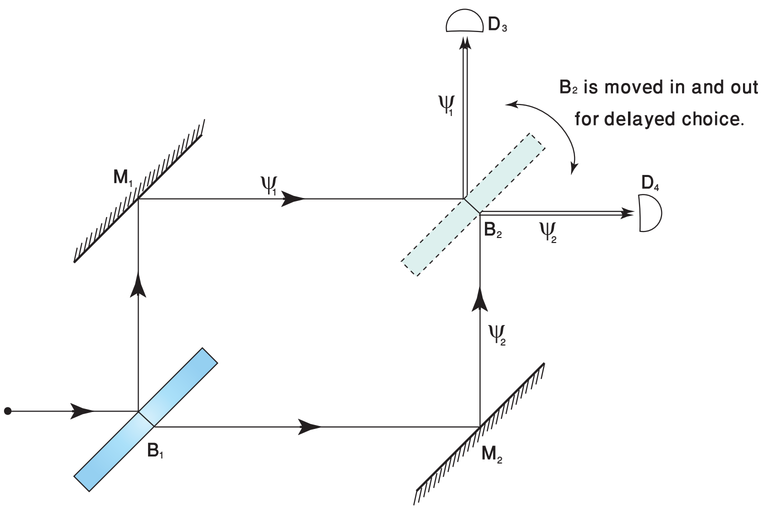


ANALOGY: A beamsplitter creates a superposition, as do the two slits in a double slit apparatus.

part (ix):

If T_1 (or T_2) clicks, the path of the single electron is determined. We have therefore chosen one path in the realm of possibilities. The momentum of the electron becomes uncertain and interference cannot therefore take place. So D_3 and D_4 each click with a 50% probability. Note that unlike a photon, the electron is not destroyed. Hence even if T_2 (or T_1) clicks, the electron can continue its onward journey towards B_2

part (x):



B_2 is moved in and out after the photon has interacted with B_1. If B_2 is in place, D_3 and D_4 will click with probabilities 1 and 0, while if B_2 is removed, D_3 and D_4 will click with 50% probability.

Answer 2:

(a) We are given that,

$$\begin{aligned}\text{Radius of hydrogen atom} &= r = 0.1 \text{ nm} = 0.1 \times 10^{-9} \text{ m} \\ \text{Uncertainty in electron's position} &= \Delta r \sim 0.1 \times 10^{-9} \text{ m.}\end{aligned}$$

The minimum uncertainty in momentum can be calculated by using uncertainty relation,

$$\begin{aligned}\Delta p \Delta r &\geq \frac{\hbar}{2} \\ \Rightarrow \Delta p &\sim \frac{\hbar}{2\Delta r} \\ &= \frac{1.06 \times 10^{-34} \text{ Js}}{2 \times 0.1 \times 10^{-9} \text{ m}} \\ &= 5.3 \times 10^{-25} \text{ kg m s}^{-1}.\end{aligned}$$

(b) Let the momentum be at least as big as the uncertainty $p \sim \Delta p$. Now minimum energy of the electron is,

$$\begin{aligned}E &= \sqrt{(p_e c)^2 + (m_e c^2)^2} \\ &= c\sqrt{p_e^2 + m_e^2 c^2} \\ &= 3.0 \times 10^8 \text{ ms}^{-1} \sqrt{(5.3 \times 10^{-25} \text{ kgms}^{-1})^2 + (9.11 \times 10^{-31} \text{ kg})^2 (3.0 \times 10^8 \text{ ms}^{-1})^2} \\ &= 8.2 \times 10^{-14} \text{ J.}\end{aligned}$$

(c) Now the minimum kinetic energy of electron will be the difference of its total and rest mass energy, i.e.,

$$\begin{aligned}K &= E - m_e c^2 \\ &= 8.2 \times 10^{-14} \text{ J} - (9.11 \times 10^{-31} \text{ kg} \times (3.0 \times 10^8 \text{ ms}^{-1})^2) \\ &= 1.0 \times 10^{-17} \text{ J} \\ &= 62.5 \text{ eV.}\end{aligned}$$

(d) The electron will be bound if this kinetic energy is larger in magnitude to the potential energy of the electron, due to electrostatic attraction, which is,

$$\begin{aligned} |V| &= \frac{ke^2}{r} \\ &= \frac{9 \times 10^9 \text{ Nm}^2\text{C}^{-2}(1.6 \times 10^{-19} \text{ C})^2}{0.1 \times 10^{-9} \text{ m}} \\ &= 2.3 \times 10^{-18} \text{ J} \\ &= \frac{2.3 \times 10^{-18}}{1.6 \times 10^{-19}} \text{ eV} \\ &\approx 14.4 \text{ eV}. \end{aligned}$$

Since $K > |V|$, the electron cannot be confined to this small radius. Such a hypothetical atom will be unstable.