Solution Tutorial 5: Modern Physics

1. An electron is confined in an infinite well of 30 cm width.

   (a) What is the ground-state energy?

   (b) In this state, what is the probability that the electron would be found within 10 cm of the left-hand wall?

   (c) If the electron instead has an energy of 1.0 eV, what is the probability that it would be found within 10 cm of the left-hand wall?

   (d) For the 1-eV electron, what is the distance between nodes and the minimum possible fractional decrease in energy?

   Answer 1:

   (a) For an infinite square well, the energy is,

$$E_n = \frac{\pi^2\hbar^2 n^2}{2mL^2}.$$

For the ground state, $n = 1$ and the corresponding energy is,

$$\begin{aligned} E_1 &= \frac{\pi^2\hbar^2}{2mL^2} \\ &= \frac{\pi^2(1.054 \times 10^{-34}\,\text{J sec})^2}{2(9.1 \times 10^{-31}\,\text{kg})(0.3\,\text{m})^2} \\ &= 6.71 \times 10^{-37}\,\text{J}. \end{aligned}$$

(b) The wavefunction for an infinite square well is,

$$\begin{aligned} \psi_n &= \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right) \\ \psi_1 &= \sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right). \end{aligned}$$

The probability of finding the electron within 10 cm of the left-hand wall is,

$$\begin{aligned} P(0 < x < 0.1\,\text{m}) &= \int_0^{0.1} \psi^*(x)\psi(x)dx \\ &= \int_0^{0.1} |\psi(x)|^2 dx \\ &= \frac{2}{L}\int_0^{0.1} \sin^2(\pi x/L)dx \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{L}\int_0^{0.1} \frac{(1-\cos(2\pi x/L))}{2}dx \\
&= \frac{1}{L}\int_0^{0.1} (1-\cos(2\pi x/L))dx \\
&= \frac{1}{L}\left[1-\frac{\sin(2\pi x/L)}{2\pi/L}\right]_0^{0.1} \\
&= 0.21.
\end{aligned}$$

(c) If the electron has 1.0 eV of energy, then,

$$\begin{aligned}
\frac{\pi^2\hbar^2 n^2}{2mL^2} &= 1.6\times 10^{-19}\,\mathrm{J} \\
n^2 &= \frac{2mL^2(1.6\times 10^{-19}\,\mathrm{J})}{\pi^2\hbar^2} \\
&= \frac{2(9.11\times 10^{-31}\,\mathrm{kg})(0.3\,\mathrm{m})^2(1.6\times 10^{-19}\,\mathrm{J})}{\pi^2(1.054\times 10^{-34}\,\mathrm{J\ sec})^2} \\
&= 2.38\times 10^{17} \\
n &= 4.88\times 10^{8}.
\end{aligned}$$

With this energy of electron, the probability of finding it within 10 cm of left-hand wall is,

$$\begin{aligned}
P(0<x<0.1\,\mathrm{m}) &= \int_0^{0.1} \psi^*(x)\psi(x)dx \\
&= \frac{2}{L}\int_0^{0.1} \sin^2(n\pi x/L)dx \\
&= \frac{2}{L}\int_0^{0.1} \frac{(1-\cos(2n\pi x/L))}{2}dx \\
&= \frac{1}{L}\int_0^{0.1} (1-\cos(2n\pi x/L))dx \\
&= \frac{1}{L}\left[1-\frac{\sin(2n\pi x/L)}{2n\pi/L}\right]_0^{0.1} \\
&= 0.33.
\end{aligned}$$

(d) We know that,

$$\begin{aligned}
L &= n\frac{\lambda}{2} \\
\lambda &= \frac{2L}{n}.
\end{aligned}$$

Now the distance between the nodes is,

$$\begin{aligned}
\frac{\lambda}{2} &= \frac{L}{2} = \frac{0.3}{4.8\times 10^8} \\
&= 6.15\,\mathring{A}.
\end{aligned}$$

The maximum possible fractional decrease in energy is thus,

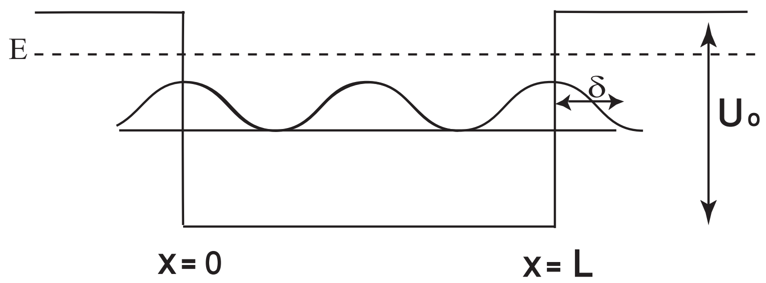
$$\begin{aligned}\frac{\Delta E}{E} &= \frac{E_n - E_{n-1}}{E_n} \\ &= \frac{n^2 - (n-1)^2}{n^2} \\ &= \frac{2}{n} - \frac{1}{n^2},\end{aligned}$$

since $n = 4.8 \times 10^8$, the minimum fractional decrease in energy is,

$$\frac{\Delta E}{E} = 4.1 \times 10^{-9}\,\text{J}.$$

2. A 50 eV electron is trapped in a finite well. How "far" (in eV) is it from being free if the penetration length of its wave function into the classically forbidden region is 1 nm?

Answer 2:



The penetration depth δ is given by,

$$\begin{aligned}\delta &= \frac{\hbar}{\sqrt{2m(U_0 - E)}} = 1 \times 10^{-9}\,\text{m} \\ 2m(U_0 - E) &= \frac{\hbar^2}{(1 \times 10^{-9}\,\text{m})^2} \\ U_0 - E &= \frac{\hbar^2}{2m(1 \times 10^{-9}\,\text{m})^2} \\ &= 38.2\,\text{meV}.\end{aligned}$$