# Solution Tutorial 8

**Q1** The $1s$ wavefunction is given as,

$$\psi_{n=1,l=0,m_l=0}(r,\theta,\phi) = \frac{1}{\sqrt{\pi}a_o^{3/2}} e^{-r/a_o}.$$

The radial probability density can be found out using the following relationship,

$$P_r(r) = r^2|R(r)|^2.$$

The ratio of these densities at $a_o$ and $a_o/2$ is,

$$\begin{aligned}
\frac{P_r(a_o)}{P_r(a_o/2)} &= \frac{a_o^2|R(a_o)|^2}{(a_o/2)^2|R(a_o/2)|^2}, \\
&= \frac{a_o^2(e^{-a_o/a_o})^2}{(a_o/2)^2(e^{-a_o/2a_o})^2}, \\
&= \frac{4\,(e^{-1})^2}{(e^{-1/2})^2}, \\
&= \frac{4}{e} = 1.47.
\end{aligned}$$

The electron is 47 % more likely to be found at $a_o$ than at $a_o/2$.

**Q2** a) The potential energy of a spherical infinite well is given as,

$$U(r) = \begin{cases} 0 & r < a \\ \infty & r > a \end{cases}$$

The radial part of the Schrodinger equation in the region $r < a$ is,

$$-\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)R(r) + \frac{\hbar^2 l(l+1)}{2mr^2}R(r) = ER(r). \qquad (5)$$

The proposed solution is,

$$R(r) = A\frac{\sin(br)}{r}.$$

Differentiating w.r.t $r$ yields,

$$\frac{d}{dR}R(r) = A\left[-\frac{\sin(br)}{r^2} + \frac{b}{r}\cos(br)\right],$$

and mutiplying $r^2$ on both sides yields,

$$r^2\frac{d}{dR}R(r) = A\left[-\sin(br) + br\cos(br)\right].$$

Differentiating once again,

$$\begin{aligned}\frac{d}{dr}\left(r^2\frac{d}{dR}\right)R(r) &= A(-b\cos(br) + b\cos(br) - b^2 r\sin(br)),\\ &= -Ab^2 r\sin(br).\end{aligned}$$

The L.H.S of Equation (5) becomes,

$$\begin{aligned}\text{L.H.S} &= -\frac{\hbar^2}{2m}\frac{1}{r^2}(-Ab^2 r\sin(br)) + \frac{\hbar^2 l(l+1)}{2mr^2}A\frac{\sin(br)}{r},\\ &= \frac{A\hbar^2}{2mr^2}\left(b^2 r\sin(br) + \frac{l(l+1)\sin(br)}{r}\right),\end{aligned}$$

and the,

$$\text{R.H.S} = EA\,\frac{\sin(br)}{r}.$$

Comparing L.H.S and R.H.S,

$$\frac{A\hbar^2}{2mr^2}\left(b^2 r\sin(br) + \frac{l(l+1)\sin(br)}{r}\right) = EA\,\frac{\sin(br)}{r}.$$

Canceling $A\sin(br)/r$ from both sides, results in the expression,

$$\frac{\hbar^2}{2m}\left(b^2 + \frac{l(l+1)}{r^2}\right) = E. \tag{6}$$

Since the above equation must hold true for all $r$, the only possibility is that we set $l = 0$, yielding,

$$E = \frac{\hbar^2 b^2}{2m}.$$

Hence $A\sin(br)/r$ is indeed a solution provided $l = 0$, therefore the orbital angular momentum must be zero.

(b) The potential is infinite, so $\psi$ must disappear at $r \geq a$. Ensuring continuity,

$$\psi(a) = A\,\frac{\sin(ba)}{a} = 0,$$

$$\Rightarrow ba = n\pi \Rightarrow b = \frac{n\pi}{a}, \quad \text{where } n \text{ is an integer.}$$

Hence the quantized energies are,

$$E_n = \frac{\hbar^2 n^2 \pi^2}{2ma^2}.$$