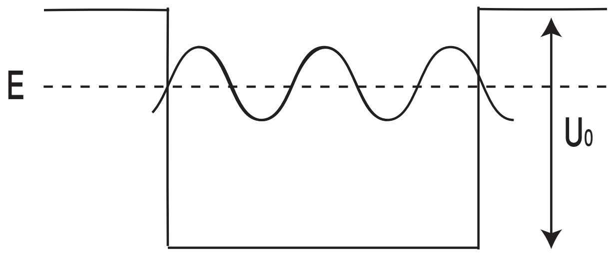
Solution Tutorial 7: Modern Physics

1. While an infinite well has an infinite number of bound states, a finite well does not. By relating the well height U_0 to the kinetic energy, and the kinetic energy (through λ) to n and L, show that the number of bound states is given roughly by $\sqrt{8mL^2U_0/h2}$.

   Answer 1:

   Refer to the Figure above. We assume that the wavefunction and energies correspond to the infinite well. If $E < U_0$, then energy is totally kinetic inside the well and is given by,

$$\frac{\hbar^2 k^2}{2m} = \frac{n^2\pi^2\hbar^2}{2mL^2}.$$

   The highest bound state exists when $E \approx U_0$. i.e., the difference $E - U_0 \longrightarrow 0$. For this state,

$$\begin{aligned} \frac{n_{\max}^2\pi^2\hbar^2}{2mL^2} &\approx U_0 \\ n_{\max} &\approx \sqrt{\frac{8U_0mL^2}{h^2}}, \end{aligned}$$

   which is the required answer.

2. A particle of mass m moves in a three-dimensional box with edge lengths L_1, L_2, L_3. Find the energies of the six lowest states if $L_1 = L$, $L_2 = 2L$, and $L_3 = 2L$. Which of these states are degenerate?

   Answer 2:

$$\begin{aligned} |p_x| &= \hbar k_x = n_1\frac{\pi\hbar}{L_1} & n_1 &= 1, 2, 3, \ldots \\ |p_y| &= \hbar k_y = n_2\frac{\pi\hbar}{L_2} & n_2 &= 1, 2, 3, \ldots \\ |p_z| &= \hbar k_z = n_3\frac{\pi\hbar}{L_3}. & n_3 &= 1, 2, 3, \ldots \end{aligned}$$

The allowed energies are,

$$\begin{aligned} E &= (|p_x|^2 + |p_x|^2 + |p_x|^2)/2m \\ &= \frac{\hbar^2\pi^2}{2m}\left[\left(\frac{n_1}{L_x}\right)^2 + \left(\frac{n_2}{L_y}\right)^2 + \left(\frac{n_3}{L_z}\right)^2\right], \end{aligned}$$

whereas we are also provided with the information, $L_x = L, L_y = L_z = 2L$. Now suppose that,

$$E_0 = \frac{\hbar^2\pi^2}{8mL^2}.$$

Substituting the given values in our energy expression we obtain,

$$\begin{aligned} E &= \frac{\hbar^2\pi^2}{2m}\left[\left(\frac{n_1^2}{L^2}\right) + \left(\frac{n_2^2}{4L^2}\right) + \left(\frac{n_3^2}{4L^2}\right)\right] \\ &= \frac{\hbar^2\pi^2}{8mL^2}\left[4n_1^2 + n_2^2 + n_3^2\right] \\ &= E_0[4n_1^2 + n_2^2 + n_3^2]. \end{aligned}$$

Choosing the quantum numbers n_1, n_2 and n_3 appropriately, helps us build the table given below.

n_1 n_2 n_3	E/E_0	comments
1 1 1	6	ground state
1 1 2	9	first excited state
1 2 1	9	first excited state
1 2 2	12	second excited state
1 3 1	14	third excited state
1 1 3	14	third excited state
2 1 1	18	fourth excited state
1 4 1	21	fifth excited state
1 1 4	21	fifth excited state
2 1 2	21	fifth excited state
2 2 1	21	fifth excited state

The table shows the lowest energy states, with relative energies,

$$\frac{E}{E_0} = 6, 9, 12, 14, 18, 21.$$

The first, third and fifth excited states are two, two and four-fold generate respectively.