Relativistic Equation of motion

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$$\cancel{\mathcal{L} = m\sqrt{1-\dot{x}^2} + q}$$

Let's start with the action for a free particle.

$$S = -m\int_{t_a}^{t_b} d\tau = -m\int_{t_a}^{t_b}\sqrt{1-\dot{x}^2}\,dt$$

where $\dot{x}^2 = |\dot{\vec{x}}|^2$.

For the interaction, let's construct a Lorentz invariant term

$$q\int_a^b dx^\mu\, A_\mu(t,\vec{x})$$

where a and b are two points (end-points) in Minkowski space.

$$S = -m\int_{t_a}^{t_b}\sqrt{1-\dot{x}^2}\,dt + q\int_a^b dx^\mu\, A_\mu(t,\vec{x})$$

$$= -m\int_{t_a}^{t_b}\sqrt{1-\dot{x}^2}\,dt + q\int_{t_a}^{t_b} dt\,\frac{dx^\mu}{dt}\,A_\mu(t,\vec{x})$$

$$\Rightarrow \mathcal{L} = -m\sqrt{1-\dot{x}^2} + q v^\mu A_\mu(t,\vec{x})\,;\quad \mu = 0,1,2,3.$$

$$= -m\sqrt{1-\dot{x}^2} + q v^0 A_0(t,\vec{x}) + q v^p A_p(t,\vec{x})$$

plus sign

Euler-Lagrange equation for the particle is :

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^p}\right) = \frac{\partial \mathcal{L}}{\partial x^p} \quad \text{for each } p = 1, 2, 3.$$

Now let's look at these terms, one by one.

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^p} = \frac{\partial}{\partial \dot{x}^p}\left(-m\sqrt{1-\dot{x}^p\dot{x}_p}\right) + q\,A_p(t,\vec{x})$$

$$= -\frac{m}{2\sqrt{1-\dot{x}^2}}\left(-\frac{\partial}{\partial \dot{x}^p}\left(\dot{x}^p\dot{x}_p\right)\right) + q\,A_p(t,\vec{x})$$

$$= \frac{m}{2\sqrt{1-\dot{x}^2}}\,\frac{\partial}{\partial \dot{x}^p}\left(\dot{x}^p(-\dot{x}^p)\right) + q\,A_p(t,\vec{x})$$

$$= -\frac{m}{\sqrt{1-\dot{x}^2}}\,\dot{x}^p + q\,A_p(t,\vec{x})$$

$$= +\frac{m}{\sqrt{1-\dot{x}^2}}\,\dot{x}_p + q\,A_p(t,\vec{x})$$

$$\text{LHS} = \frac{d}{dt}\left(\frac{m\,\dot{x}_p}{\sqrt{1-\dot{x}^2}}\right) + q\frac{d}{dt}\left(A_p(t,\vec{x})\right)$$

note the change from p to n.

$$\text{RHS} = \frac{\partial \mathcal{L}}{\partial x^p} = q\,\underline{v}^0\,\frac{\partial}{\partial x^p}A_0(t,\vec{x})$$

$$+\ q\,v^n\,\frac{\partial}{\partial x^p}A_n(t,\vec{x})$$

$$RHS = q\, v^0 \frac{\partial}{\partial x^p} A_0 + q\, v^n \frac{\partial}{\partial x^p} A_n$$

(I am suppressing the argument $(t, \vec{x})$ on the R.H.S for the time being).

Hence

$$\frac{d}{dt}\left(\frac{m\, \dot{x}_p}{\sqrt{1-\dot{x}^2}}\right) + q \frac{d}{dt}\left(A_p(t, \vec{x})\right)$$
$$= q\, v^0 \frac{\partial}{\partial x^p} A_0 + q\, v^n \frac{\partial}{\partial x^p} A_n$$

Investigate $\frac{d}{dt}\left(A_p(t, \vec{x})\right)$ a bit closely.

$$\frac{d}{dt}\left(A_p(t, \vec{x})\right) = \frac{\partial A_p}{\partial t} + \dot{x}^n \frac{\partial A_p}{\partial x^n}$$

$n = 1, 2, 3.$

$\therefore$

$$\frac{d}{dt}\left(\frac{m\, \dot{x}_p}{\sqrt{1-\dot{x}^2}}\right) + q \frac{\partial A_p}{\partial t} + q\, \dot{x}^n \frac{\partial A_p}{\partial x^n}$$
$$= q\, v^0 \frac{\partial}{\partial x^p} A_0 + q\, v^n \frac{\partial A_n}{\partial x^p}$$

$$\boxed{\frac{dx^0}{dt} = \frac{dt}{dt} = 1}$$

$$\frac{d}{dt}\left(\frac{m\,\dot{x}_p}{\sqrt{1-\dot{x}^2}}\right) = q\,v^0\,\frac{\partial A_0}{\partial x^p} - q\,\frac{\partial A_p}{\partial t} + q\,\dot{x}^n\left(\frac{\partial A_n}{\partial x^p} - \frac{\partial A_p}{\partial x^n}\right)$$

$$\boxed{\frac{d}{dt}\left(\frac{m\,\dot{x}_p}{\sqrt{1-\dot{x}^2}}\right) = q\left(\frac{\partial A_0}{\partial x^p} - \frac{\partial A_p}{\partial t}\right) + q\,\dot{x}^n\left(\frac{\partial A_n}{\partial x^p} - \frac{\partial A_p}{\partial x^n}\right)} \quad \boxed{\text{Eq. } (1)}$$

Define $\frac{\partial A_0}{\partial x^p} - \frac{\partial A_p}{\partial t} = \frac{\partial A_0}{\partial x^p} - \frac{\partial A_p}{\partial x^0} = E_p$

e.g. $E_x = \frac{\partial A_0}{\partial x} - \frac{\partial A_x}{\partial t}$ etc. where $A_0 = -\phi$

and ϕ is the conventional electric potential.

Let's identify the second term on the RHS.

Now

$$\left(\vec{v} \times \vec{B}\right)_p = \epsilon_{pqr}\, v^q B_r$$

If $\vec{B} = \vec{\nabla} \times \vec{A}$ and $B_r = \epsilon_{rst}\frac{\partial}{\partial x^s} A_t$ (Def. of curl in tensor notation).

then $\left(\vec{v} \times \vec{B}\right)_p = \epsilon_{pqr}\, v^q\, \epsilon_{rst}\,\frac{\partial A_t}{\partial x^s}$

$$(\vec{V} \times \vec{B})_p = \epsilon_{pqr}\,\epsilon_{rst}\, v^q \frac{\partial A_t}{\partial x^s}$$

$$= \epsilon_{rpq}\,\epsilon_{rst}\, v^q \frac{\partial A_t}{\partial x^s}$$

$$= \left(\delta_{ps}\,\underbrace{\delta_{qt}} - \delta_{pt}\,\delta_{qs}\right) v^q \frac{\partial A_t}{\partial x^s}$$

(using an identity involving Levi-Civita symbols ↓ refer to your electrodynamics course)

$$= v^t \frac{\partial A_t}{\partial x^p} - v^q \frac{\partial A_p}{\partial x^q}$$

$$= v^t \frac{\partial A_t}{\partial x^p} - v^t \frac{\partial A_p}{\partial x^t}$$

$$= v^n \frac{\partial A_n}{\partial x^p} - v^n \frac{\partial A_p}{\partial x^n}$$

(change variables in the repeated sum)

$$= \dot{x}^n \left(\frac{\partial A_n}{\partial x^p} - \frac{\partial A_p}{\partial x^n}\right)$$

which is <u>identical</u> to the second term on the RHS.

Hence

$$\boxed{\frac{d}{dt}\left(\frac{m\,\dot{x}_p}{\sqrt{1-\dot{x}^2}}\right) = q\,E_p + q\left(\vec{V} \times \vec{B}\right)_p}$$

which is the correct relativistic equation for the 3-vector. Let's cast this in a manifestly Lorentz invariant form.

Writing a manifestly Lorentz invariant form

$$\frac{d}{dt}\left(m\, u_p\right) = q\left(\frac{\partial A_0}{\partial x^p} - \frac{\partial A_p}{\partial x^0}\right) + q\,\dot{x}^n\left(\frac{\partial A_n}{\partial x^p} - \frac{\partial A_p}{\partial x^n}\right)$$

Multiply by $\frac{dt}{d\tau} = \frac{dx^0}{d\tau}$.

$$\boxed{\begin{aligned} n &= 1,2,3 \\ p &= 1,2,3 \end{aligned}}$$

$$\frac{d}{d\tau}\left(m\, u_p\right) = q\,\frac{dx^0}{d\tau}\frac{\partial A_0}{\partial x^p} - q\,\frac{dx^0}{d\tau}\frac{\partial A_p}{\partial x^0} + q\,\dot{x}^n\frac{dx^0}{d\tau}\frac{\partial A_n}{\partial x^p} - q\,\dot{x}^n\frac{dx^0}{d\tau}\frac{\partial A_p}{\partial x^n}$$

Combine

Combine

$$= q\left(\frac{dx^0}{d\tau}\frac{\partial A_0}{\partial x^p} + \frac{dx^n}{d\tau}\frac{\partial A_n}{\partial x^p}\right)$$

$$- q\left(\frac{dx^0}{d\tau}\frac{\partial A_p}{\partial x^0} + \frac{dx^n}{d\tau}\frac{\partial A_p}{\partial x^n}\right)$$

$$= q\left(\frac{dx^\mu}{dt}\frac{\partial A_\mu}{\partial x^p}\right) - q\left(\frac{dx^\mu}{d\tau}\frac{\partial A_p}{\partial x^\mu}\right)$$

where $\mu = \underline{\underline{0}}, 1, 2, 3$.

added

Hence

$$\frac{d}{d\tau}\left(m\, u_p\right) = q\,\frac{dx^\mu}{d\tau}\left(\frac{\partial A_\mu}{\partial x^p} - \frac{\partial A_p}{\partial x^\mu}\right)$$

p is a free index and $p = 1, 2, 3$. μ is a contracted index.

Let's replace μ by ν, no harm done.

$$\boxed{\frac{d}{d\tau}\left(m u_\rho\right) = q\,\frac{dx^\nu}{d\tau}\left(\frac{\partial A_\nu}{\partial x^\rho} - \frac{\partial A_\rho}{\partial x^\nu}\right)} \quad \text{— (2)}$$

Now S is Lorentz invariant. Now If the three spatial Equations of motion are Lorentz invariant equations of motion derived from $\mathcal{L}$ (a Lorentz invariant Lagrangian) give three components of a 3-vector, each one being equal to the three components of another 3-vector (Eq. (1)), then the time component must also be equal.

Hence if I set the zero'th component in the left and R.H.S. of Eq. (2), then it should be a valid equality.

i.e

$$\frac{d}{d\tau}\left(m u_0\right) = q\,\frac{dx^\nu}{d\tau}\left(\frac{\partial A_\nu}{\partial x^0} - \frac{\partial A_0}{\partial x^\nu}\right)$$

<u>check.</u>

$$LHS = \frac{d}{dt}\left(m\frac{1}{\sqrt{1-v^2}}\right) = m\frac{d}{dt}\left(u_0\right) = \frac{d}{dt}\left(\text{energy}\right)$$

since $m u_0 = $ energy (zero'th term = the 4-velocity)

$$RHS = q\,\frac{dx^\nu}{d\tau}\left(\frac{\partial A_\nu}{\partial x^0} - \frac{\partial A_0}{\partial x^\nu}\right) \cancel{= q\frac{dx^\nu}{d\tau}E_\nu}$$

$\cancel{=}$

$$= q\left[\frac{dx^0}{d\tau}\left(\frac{\partial A_0}{\partial x^0} - \overset{\text{zero}}{\frac{\partial A_0}{\partial x^0}}\right) + \frac{dx^n}{d\tau}\left(\underbrace{\frac{\partial A_n}{\partial x^0} - \frac{\partial A_0}{\partial x^n}}\right)\right]$$

$$= q\,\frac{dx^n}{d\tau}\left(-E_n\right)$$

$$\frac{d}{d\tau}(m u_0) = -q\,\frac{dx^n}{d\tau} E_n$$

$$\frac{d\tau}{dt}\,\frac{d}{d\tau}(m u_0) = -q\,\frac{d\tau}{dt}\,\frac{dx^n}{d\tau} E_n$$

$$\frac{d}{dt}(m u_0) = -q\,\frac{dx^n}{dt} E_n$$

rate of change of K.E $= -q\,\vec{v}\cdot\vec{E}$ (relativistic work energy principle)

Key idea

$\mathcal{L} \rightarrow$ Lorentz invariant

$\Rightarrow$ equations of motion are Lorentz invariant

$\Rightarrow$ space 3-vector component equations are formed

$\Rightarrow$ the time component of the equation of motion will follow.

Because of the Lorentz invariance of the equations of motion, Eq. (2) will also hold if I replaced p_i with another Greek free index μ which can take up the value 0, 1, 2, 3. We have already verified it for $\mu = 1, 2, 3$ (Eq. (2)) and $\mu = 0$ follows because of Lorentz invariance of the overall motion. Besides, it makes sense intuitively as it reproduces the relativistic work-energy principle.