Solution Assignment 1: Quantum Field Theory

1. $(x^\mu + y^\mu)(x_\mu + y_\mu)$ being a scalar is Lorentz invariant. Now

$$(x^\mu + y^\mu)(x_\mu + y_\mu) = x^\mu x_\mu + x^\mu y_\mu + y^\mu x_\mu + y^\mu y_\mu \tag{1}$$

Similarly $(x^\mu - y^\mu)(x_\mu - y_\mu)$ is also Lorentz invariant.

$$(x^\mu - y^\mu)(x_\mu - y_\mu) = x^\mu x_\mu - x^\mu y_\mu - y^\mu x_\mu + y^\mu y_\mu \tag{2}$$

Subtracting equation (1) and (2) yields yet another Lorentz invariant.

$$\begin{aligned}
& (x^\mu + y^\mu)(x_\mu + y_\mu) - (x^\mu - y^\mu)(x_\mu - y_\mu) \\
= \;& (x^\mu x_\mu + x^\mu y_\mu + y^\mu x_\mu + y^\mu y_\mu) - (x^\mu x_\mu - x^\mu y_\mu - y^\mu x_\mu + y^\mu y_\mu) \\
= \;& 2(x^\mu y_\mu + x_\mu y^\mu) \\
= \;& 4x^\mu y_\mu \qquad \text{since} \quad a^\mu b_\mu = a_\mu b^\mu
\end{aligned}$$

This implies that $x^\mu y_\mu$ is also Lorentz invariant.

2.

$$\mathscr{L}' = \frac{1}{2}(\partial_\mu \phi_1')^2 + \frac{1}{2}(\partial_\mu \phi_2')^2 - \frac{1}{2}m{\phi_1'}^2 - \frac{1}{2}m{\phi_2'}^2 + g({\phi_1'}^2 + {\phi_2'}^2)^2, \tag{3}$$

where

$$\begin{aligned}
\phi_1' &= \cos\theta\ \phi_1 - \sin\theta\ \phi_2 \\
\phi_2' &= \sin\theta\ \phi_1 + \cos\theta\ \phi_2
\end{aligned}$$

Here θ represents some internal degree of freedom.

Substituting these fields to equation (3) yields

$$\begin{aligned}
\mathscr{L}' =\;& \frac{1}{2}\Big[\partial_\mu(\cos\theta\ \phi_1 - \sin\theta\ \phi_2)\Big]^2 + \frac{1}{2}\Big[\partial_\mu(\sin\theta\ \phi_1 + \cos\theta\ \phi_2)\Big]^2 - \frac{1}{2}m(\cos\theta\ \phi_1 - \sin\theta\ \phi_2)^2 \\
-\;& \frac{1}{2}m(\sin\theta\ \phi_1 + \cos\theta\ \phi_2)^2 + g\Big[(\cos\theta\ \phi_1 - \sin\theta\ \phi_2)^2 + (\sin\theta\ \phi_1 + \cos\theta\ \phi_2)^2\Big]^2 \\
=\;& \frac{1}{2}\Big[(\partial_\mu \cos\theta\ \phi_1)^2 + (\partial_\mu \sin\theta\ \phi_2)^2 - 2\partial_\mu(\cos\theta\ \phi_1)\partial_\mu(\sin\theta\ \phi_2)\Big] \\
+\;& \frac{1}{2}\Big[(\partial_\mu \sin\theta\ \phi_1)^2 + (\partial_\mu \cos\theta\ \phi_2)^2 + 2\partial_\mu(\sin\theta\ \phi_1)\partial_\mu(\cos\theta\ \phi_2)\Big] \\
-\;& \frac{1}{2}m(\cos^2\theta\ \phi_1^2 + \sin^2\theta\ \phi_2^2 - 2\cos\theta\sin\theta\ \phi_1\phi_2) - \frac{1}{2}m(\sin^2\theta\ \phi_1^2 + \cos^2\theta\ \phi_2^2 + 2\cos\theta\sin\theta\ \phi_1\phi_2) \\
+\;& g(\cos^2\theta\ \phi_1^2 + \sin^2\theta\ \phi_2^2 - 2\cos\theta\sin\theta\ \phi_1\phi_2 + \sin^2\theta\ \phi_1^2 + \cos^2\theta\ \phi_2^2 + 2\cos\theta\sin\theta\ \phi_1\phi_2)^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\big[\cos^2\theta(\partial_\mu\phi_1)^2 + \sin^2\theta(\partial_\mu\phi_2)^2 - 2\partial_\mu(\cos\theta\phi_1)\partial_\mu(\sin\theta\phi_2)\\
&+ \sin^2\theta(\partial_\mu\phi_1)^2 + \cos^2\theta(\partial_\mu\phi_2)^2 + 2\partial_\mu(\sin\theta\phi_1)\partial_\mu(\cos\theta\phi_2)\big]\\
&- \frac{1}{2}m\big[\cos^2\theta\ \phi_1^2 + \sin^2\theta\ \phi_2^2 - 2\cos\theta\sin\theta\ \phi_1\phi_2 + \sin^2\theta\ \phi_1^2 + \cos^2\theta\ \phi_2^2 + 2\cos\theta\sin\theta\ \phi_1\phi_2\big]\\
&+ g\big[(\cos^2\theta + \sin^2\theta)\phi_1^2 + (\cos^2\theta + \sin^2\theta)\phi_2^2\big]^2\\
&= \frac{1}{2}\big[(\partial_\mu\phi_1)^2 + (\partial_\mu\phi_2)^2\big] - \frac{1}{2}m(\phi_1^2 + \phi_2^2) + g(\phi_1^2 + \phi_2^2)^2\\
&= \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 - \frac{1}{2}m\phi_1^2 - \frac{1}{2}m\phi_2^2 + g(\phi_1^2 + \phi_2^2)^2\\
&= \mathscr{L}.
\end{aligned}
$$

Hence the Lagrangian is invariant under the SO(2) transformation.

3. For a particle

$$
\begin{aligned}
\frac{\partial\mathscr{L}}{\partial x} &= -\frac{\partial\phi(x)}{\partial x}\sqrt{1-\dot{x}^2}\\
\frac{\partial\mathscr{L}}{\partial(\dot{x})} &= -\frac{m+\phi(x)}{2\sqrt{1-\dot{x}^2}}(-2\dot{x}) = -\frac{m+\phi(x)}{\sqrt{1-\dot{x}^2}}\dot{x}\\
\frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial(\dot{x})}\right) &= -\big(m+\phi(x)\big)\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-\dot{x}^2}}\right) = -\frac{\partial\phi(x)}{\partial x}\sqrt{1-\dot{x}^2}\ .
\end{aligned}
$$

4.

$$
\text{Now}\quad A'_\mu = g_\mu{}^\nu A'^\nu = g_{\mu\nu}\Lambda_w{}^\nu A^w \tag{4}
$$

$$
\text{But}\quad A'_\mu = M_\mu{}^\tau A_\tau \tag{5}
$$

Comparing equation (4) and (5)

$$
\begin{aligned}
M_\mu{}^\tau A_\tau &= g_{\mu\nu}\Lambda_w{}^\nu A^w\\
&= g_{\mu\nu}\Lambda_w{}^\nu g^{w\lambda}A_\lambda\\
&= g_{\mu\nu}\Lambda_w{}^\nu g^{w\tau}A_\tau\\
\text{Hence}\qquad M_\mu{}^\tau &= g_{\mu\nu}\Lambda_w{}^\nu g^{w\tau}.
\end{aligned}
$$

is the required relationship.

5. With the given Lagrangian,

$$\begin{aligned}
\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} &= \partial_\mu \phi \\
\frac{\partial \mathscr{L}}{\partial \phi} &= -m^2\phi - \sum_{n=1}^{\infty} \lambda_n (2n+2)\phi^{2n+1} \\
\partial^2 \phi &+ m^2\phi + \sum_{n=1}^{\infty} \lambda_n (2n+2)\phi^{2n+1} = 0 \\
(\partial^2 &+ m^2)\phi + \sum_{n=1}^{\infty} \lambda_n (2n+2)\phi^{2n+1} = 0
\end{aligned}$$

is the desired equation of motion.

6. The simplified form of the given Lagrangian is

$$\mathscr{L} = (\partial^\mu \psi^\dagger)(\partial_\mu \psi) - m^2 \psi^\dagger \psi - g'(\psi^\dagger \psi)^2$$

upto some scalar factor.

$$\psi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)\ , \qquad \psi_2 = \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

Adding and subtracting these equations to get

$$\phi_1 = \frac{\psi + \psi^\dagger}{\sqrt{2}}\ ; \qquad \phi_2 = -i\frac{\psi - \psi^\dagger}{\sqrt{2}}$$

$$\begin{aligned}
\mathscr{L} &= \frac{1}{2}(\partial^\mu \phi_1 \partial_\mu \phi_1) + \frac{1}{2}(\partial^\mu \phi_2 \partial_\mu \phi_2) - \frac{1}{2}m^2\phi_1^2 - \frac{1}{2}m^2\phi_2^2 - g(\phi_1^2 + \phi_2^2)^2 \\
\mathscr{L} &= \frac{1}{2}\left[\partial^\mu\left(\frac{\psi + \psi^\dagger}{\sqrt{2}}\right)\partial_\mu\left(\frac{\psi + \psi^\dagger}{\sqrt{2}}\right)\right] + \frac{1}{2}\left[\partial^\mu\left(\frac{-i(\psi - \psi^\dagger)}{\sqrt{2}}\right)\partial_\mu\left(\frac{-i(\psi - \psi^\dagger)}{\sqrt{2}}\right)\right] \\
&- \frac{1}{2}m^2\left(\frac{\psi + \psi^\dagger}{\sqrt{2}}\right)^2 - \frac{1}{2}m^2\left(\frac{-i(\psi - \psi^\dagger)}{\sqrt{2}}\right)^2 - g\left[\left(\frac{\psi + \psi^\dagger}{\sqrt{2}}\right)^2 + \left(\frac{-i(\psi - \psi^\dagger)}{\sqrt{2}}\right)^2\right]^2 \\
\mathscr{L} &= \frac{1}{4}\left[(\partial^\mu \psi + \partial^\mu \psi^\dagger)(\partial_\mu \psi + \partial_\mu \psi^\dagger)\right] - \frac{1}{4}\left[(\partial^\mu \psi - \partial^\mu \psi^\dagger)(\partial_\mu \psi - \partial_\mu \psi^\dagger)\right] \\
&- \frac{1}{4}m^2(\psi^2 + {\psi^\dagger}^2 + 2\psi\psi^\dagger) + \frac{1}{4}m^2(\psi^2 + {\psi^\dagger}^2 - 2\psi\psi^\dagger) \\
&- \frac{1}{4}g(\psi^2 + {\psi^\dagger}^2 + 2\psi\psi^\dagger - \psi^2 - {\psi^\dagger}^2 + 2\psi\psi^\dagger)^2 \\
\mathscr{L} &= \frac{1}{4}(4\partial^\mu \psi^\dagger \partial_\mu \psi) - \frac{1}{4}m^2(4\psi^\dagger\psi) - 4g(\psi^\dagger\psi)^2 \\
\text{with } \ 4g &= g' \\
\mathscr{L} &= (\partial^\mu \psi^\dagger)(\partial_\mu \psi) - m^2\psi^\dagger\psi - g'(\psi^\dagger\psi)^2.
\end{aligned}$$

which is the required form.

7. (a)

$$\mathscr{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2\phi^2$$

Our definition of Hamiltonian is

$$\begin{aligned} \mathcal{H} &= \pi\dot{\phi} - \mathscr{L} \\ \text{Now} \qquad \pi &= \frac{\partial \mathscr{L}}{\partial \dot{\phi}} = \dot{\phi} \end{aligned}$$

which is called the canonical momentum.

(b) Let's find the Hamiltonian now.

$$\begin{aligned} \mathcal{H} &= \pi\dot{\phi} - \mathscr{L} \\ &= (\dot{\phi})^2 - \frac{1}{2}(\dot{\phi}^2 - \vec{\nabla}^{\,2}\phi) + \frac{1}{2}m^2\phi^2 \\ &= \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\vec{\nabla}^{\,2}\phi + \frac{1}{2}m^2\phi^2. \end{aligned}$$

(c)

$$\begin{aligned} \Pi^\mu &= \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \\ &= \partial^{\,\mu}\phi. \end{aligned}$$

$$\begin{aligned} \Pi^\circ &= \frac{\partial \mathscr{L}}{\partial(\partial_t \phi)} \\ &= \partial_t\phi = \pi. \end{aligned}$$