Solution Assignment 2: Quantum Field Theory

Note: I am using bold face math font to represent vectors.

1. **Answer** 1:

$$\partial_\nu F^{\mu\nu} = 0$$

Set $\mu = 0$

$$\begin{aligned} \partial_\nu F^{\mu\nu} &= \partial_0 F^{00} + \partial_1 F^{01} + \partial_2 F^{02} + \partial_3 F^{03} \\ &= 0 + \frac{\partial}{\partial x}\big(-E_1\big) + \frac{\partial}{\partial y}\big(-E_2\big) + \frac{\partial}{\partial z}\big(-E_3\big) = 0 \\ \Rightarrow \quad \nabla \cdot \mathbf{E} &= 0 \end{aligned}$$

Set $\mu = 1$

$$\begin{aligned} \partial_\nu F^{\mu\nu} &= \partial_\nu F^{1\nu} \\ &= \partial_0 F^{10} + \partial_1 F^{11} + \partial_2 F^{12} + \partial_3 F^{13} \\ &= \frac{\partial E_1}{\partial t} + 0 + \frac{\partial}{\partial y}\big(-B_3\big) + \frac{\partial}{\partial z}\big(B_2\big) \\ &= \frac{\partial E_1}{\partial t} - (\nabla \times \mathbf{B})_1 = 0 \\ \Rightarrow \quad \big(\nabla \times \mathbf{B}\big)_1 - \frac{\partial E_1}{\partial t} &= 0 \end{aligned}$$

Similarly setting $\mu = 2, 3$ will result in,

$$\begin{aligned} \big(\nabla \times \mathbf{B}\big)_{2,3} - \frac{\partial E_{2,3}}{\partial t} &= 0, \\ \text{which results in} \qquad (\nabla \times \mathbf{B}) - \frac{\partial \mathbf{E}}{\partial t} &= 0 \end{aligned}$$

2. **Answer** 2:

$$B_1 = B_2 = B_3 = 0$$

$$\text{Now} \qquad F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & 0 & 0 \\ -E_2 & 0 & 0 & 0 \\ -E_3 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{and} \qquad \Lambda = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We also have:

$$\begin{aligned}
(F')^{\mu\nu} &= \Lambda^{\mu}_{w}\Lambda^{\nu}_{\tau}F^{w\tau} \\
\text{e.g.} \quad {E_1}' &= (F')^{01} = \Lambda^{0}_{w}\Lambda'_{\tau}F^{w\tau} \\
&= \Lambda^{0}_{0}\Lambda^{1}_{1}F^{01} + \Lambda^{0}_{1}\Lambda^{1}_{0}F^{10} + \text{other terms which are zero} \\
&= \gamma^2 E_1 - v^2\gamma^2 E_1 = E_1. \\
\text{Similarly} \qquad E'_2 &= E'_y = (F')^{02} = \Lambda^{0}_{w}\Lambda^{2}_{\tau}F^{w\tau} \\
&= \Lambda^{0}_{w}\Lambda^{2}_{2}F^{w2} = \Lambda^{0}_{0}\Lambda^{2}_{2}F^{02} = \gamma E_2 = \frac{E_2}{\sqrt{1-v^2}}
\end{aligned}$$

Similarly,

$$\begin{aligned}
E'_3 &= \frac{E_3}{\sqrt{1-v^2}}\,. \\
B'_1 &= (F')^{23} = \Lambda^{2}_{w}\Lambda^{3}_{\tau}F^{w\tau} \\
&= \Lambda^{2}_{2}\Lambda^{3}_{3}F^{23} = 0.
\end{aligned}$$

$$
\begin{aligned}
B_2' &= -(F')^{13} = -\Lambda^1_w \Lambda^3_\tau F^{w\tau} \\
&= -\Lambda^1_0 \Lambda^3_3 F^{03} - \Lambda^1_1 \Lambda^3_3 F^{23} \\
&= (v\gamma)(1)E_3 - (\gamma)(1)(0) \\
&= \frac{v\gamma E_3}{\sqrt{1-v^2}}, \quad \text{and finally} \\
B_3' &= (F')^{12} = \Lambda^1_w \Lambda^2_\tau F^{w\tau} \\
&= \Lambda^1_0 \Lambda^2_2 F^{12} + \Lambda^1_1 \Lambda^2_2 F^{12} \\
&= -(v\gamma)(1)E_2 - (\gamma)(1)(0) \\
&= (-v\gamma E_2) \\
&= \frac{-v}{\sqrt{1-v^2}} E_2.
\end{aligned}
$$

3. **Answer** 3:

$$
\begin{aligned}
(T')^{\mu\nu} &= \Lambda^\mu_w \Lambda^\nu_\tau T^{w\tau} \\
(T')^{\nu\mu} &= \Lambda^\nu_w \Lambda^\mu_\tau T^{w\tau} \\
&= \Lambda^\nu_\tau \Lambda^\mu_w T^{\tau w} \quad \text{(merely interchange } \tau \text{ and } w \text{ indices)} \\
&= -\Lambda^\mu_w \Lambda^\nu_\tau T^{w\tau} \quad \text{since } \; T^{w\tau} = -T^{\tau w} \\
&= -(T')^{\mu\nu}.
\end{aligned}
$$

4. **Answer** 4:

$$
\mathscr{L} = -m\sqrt{1-\dot{x}^2} - mAx,
$$

where A is a constant and potential energy is given by mAx, A being a constant.

$$
\begin{aligned}
\frac{\partial \mathscr{L}}{\partial \dot{x}} &= \frac{m\dot{x}}{\sqrt{1-\dot{x}^2}} \\
\frac{\partial \mathscr{L}}{\partial x} &= -mA \\
\frac{d}{dt}\left(\frac{m\dot{x}}{\sqrt{1-\dot{x}^2}}\right) &= -mA \\
\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-\dot{x}^2}}\right) &= -A
\end{aligned}
$$

Integrate with respect to t,

$$
\frac{\dot{x}}{\sqrt{1-\dot{x}^2}} = -At + A_1,
$$

where A_1 is another constant of integration.

$$
\begin{aligned}
\sqrt{\frac{\dot{x}^2}{1-\dot{x}^2}} &= A_1 - At \\
\sqrt{\frac{1-\dot{x}^2}{\dot{x}^2}} &= \frac{1}{A_1 - At} \\
\frac{1}{\dot{x}^2} - 1 &= \frac{1}{(A_1 - At)^2} \\
\frac{1}{\dot{x}^2} &= 1 + \frac{1}{(A_1 - At)^2} \\
&= \frac{(A_1 - At)^2 + 1}{(A_1 - At)^2} \\
\dot{x}^2 &= \frac{(A_1 - At)^2}{(A_1 - At)^2 + 1} \\
\Rightarrow \quad \frac{dx}{dt} &= \frac{(A_1 - At)}{\sqrt{(A_1 - At)^2 + 1}} \\
x(t) &= \int_0^t \frac{(A_1 - At')}{\sqrt{(A_1 - At')^2 + 1}}\, dt' \\
&= \frac{1}{2A}\left(A_1^2 - (A_1 - At)^2\right)
\end{aligned}
$$

is the required trajectory.