Solution Assignment 3: Quantum Field Theory

1.

$$
\begin{aligned}
L.H.S &= \partial_\mu F_{\nu\tau} + \partial_\nu F_{\tau\mu} + \partial_\tau F_{\mu\nu} \\
&= \partial_\mu(\partial_\nu A_\tau - \partial_\tau A_\nu) + \partial_\nu(\partial_\tau A_\mu - \partial_\mu A_\tau) + \partial_\tau(\partial_\mu A_\nu - \partial_\nu A_\mu) \\
&= \partial_\mu\partial_\nu A_\tau - \partial_\mu\partial_\tau A_\nu + \partial_\nu\partial_\tau A_\mu - \partial_\nu\partial_\mu A_\tau + \partial_\tau\partial_\mu A_\nu - \partial_\tau\partial_\nu A_\mu \\
&= 0.
\end{aligned}
$$

Since $\partial_\mu\partial_\nu A_\tau = \partial_\nu\partial_\mu A_\tau$ i.e., ($T_{\mu\nu} = \partial_\mu\partial_\nu$ is symmetric).

2.

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right) &= \frac{\partial\mathscr{L}}{\partial\phi} \\
\frac{d}{dt}\Pi^\mu &= \frac{\partial\mathscr{L}}{\partial\phi} \\
\dot{\Pi}^\mu &= \frac{\partial\mathscr{L}}{\partial\phi}
\end{aligned}
$$

Hence the time derivative of Π^μ will be zero when $\mathscr{L}$ does not depend on the field ϕ.

3. (a) Tensor is,

$$
\begin{aligned}
T^{\mu\nu} &= \Pi^\mu\partial^\nu\phi - g^{\mu\nu}\mathscr{L} \\
\Pi^\mu &= \frac{\partial\mathscr{L}}{\partial_\mu\phi} = \partial_\mu\phi \\
\therefore\ T^{\mu\nu} &= (\partial_\mu\phi)(\partial^\nu\phi) - g^{\mu\nu}\left(\frac{1}{2}(\partial_\lambda\phi)^2 - \frac{1}{2}m^2\phi^2\right).
\end{aligned}
$$

that $(\partial_\lambda\phi)^2$ is short hand for $(\partial^\lambda\phi)(\partial_\lambda\phi)$. See part (b) below.

(b)

$$
\begin{aligned}
T^{00} &= (\partial_0\phi)(\partial^0\phi) - \frac{1}{2}(\partial_\lambda\phi)^2 + \frac{1}{2}m^2\phi^2 \\
&= \frac{1}{2}\left[\left(\frac{\partial\phi}{\partial t}\right)^2 + \left(\vec{\nabla}\phi\right)^2\right] + \frac{1}{2}m^2\phi^2 \\
&= \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2 = \mathcal{H}.
\end{aligned}
$$

In this calculation, we have use the formula,

$$(\partial_i\phi)(\partial^i\phi) = -(\partial_i\phi)^2.$$

(c) See the index acrobites here. I have laid out each step carefully.

$$
\begin{aligned}
\partial_\mu T^{\mu\nu} &= \partial_\mu(\partial^\mu\phi)(\partial^\nu\phi) - g^{\mu\nu}\partial_\mu\left(\frac{1}{2}(\partial_\lambda\phi)^2 - \frac{1}{2}m^2\phi^2\right) \\
&= \partial_\mu(\partial^\mu\phi)(\partial^\nu\phi) - g^{\mu\nu}\partial_\mu\left(\frac{1}{2}(\partial_\lambda\phi)^2\right) + g^{\mu\nu}\partial_\mu\left(\frac{1}{2}m^2\phi^2\right) \\
&= (\partial_\mu\partial^\mu\phi)(\partial^\nu\phi) + (\partial^\mu\phi)(\partial_\mu\partial^\nu\phi) - \partial^\nu\left(\frac{1}{2}(\partial_\lambda\phi)^2\right) + \partial^\nu\left(\frac{1}{2}m^2\phi^2\right) \\
&= (\partial^2\phi)(\partial^\nu\phi) + (\partial^\mu\phi)(\partial_\mu\partial^\nu\phi) - \frac{1}{2}\left(\partial^\nu(\partial_\lambda\phi)^2\right) + \frac{1}{2}m^2\partial^\nu(\phi^2) \\
&= (\partial^2\phi)(\partial^\nu\phi) + (\partial^\mu\phi)(\partial_\mu\partial^\nu\phi) - \frac{1}{2}\big(\partial^\nu(\partial^\lambda\phi)(\partial_\lambda\phi)\big) + \frac{1}{2}m^2\phi\partial^\nu\phi \\
&= (\partial^2\phi)(\partial^\nu\phi) + (\partial^\mu\phi)(\partial_\mu\partial^\nu\phi) - \frac{1}{2}\left[(\partial^\nu\partial^\lambda\phi)(\partial_\lambda\phi) + (\partial^\lambda\phi)(\partial^\nu\partial_\lambda\phi) + m^2\phi\partial^\nu\phi\right] \\
&= (\partial^2 + m^2)\phi(\partial^\nu\phi) + \left[(\partial^\mu\phi)(\partial_\mu\partial^\nu\phi) - \frac{1}{2}(\partial^\nu\partial^\lambda\phi)(\partial_\lambda\phi) - \frac{1}{2}(\partial^\lambda\phi)(\partial^\nu\partial_\lambda\phi)\right].
\end{aligned}
$$

first term vanishes because of Klein-Gorden equation.

$$
\begin{aligned}
\partial_\mu T^{\mu\nu} &= (\partial^\mu\phi)(\partial_\mu\partial^\nu\phi) - \frac{1}{2}(\partial^\nu\partial_\lambda\phi)(\partial^\lambda\phi) - \frac{1}{2}(\partial^\lambda\phi)(\partial^\nu\partial_\lambda\phi) \\
&= (\partial^\mu\phi)(\partial_\mu\partial^\nu\phi) - (\partial^\nu\partial_\lambda\phi)(\partial^\lambda\phi) \\
&= (\partial^\mu\phi)(\partial_\mu\partial^\nu\phi) - (\partial^\mu\phi)(\partial^\nu\partial_\mu\phi) \\
&= (\partial^\mu\phi)(\partial_\mu\partial^\nu\phi) - (\partial^\mu\phi)(\partial^\mu\partial_\nu\phi) \\
&= 0
\end{aligned}
$$

since $a^\mu b_\nu = a^\nu b_\mu$. Hence T satisfies the continuity equation.

(d)

$$
\begin{aligned}
p^\nu &= \int d^3x T^{0\nu} \\
p^0 &= \int d^3x T^{00} = \int d^3x \mathcal{H} \quad \text{is the energy inside the field} \\
&= \int d^3x\left(\frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{2}m^2\phi^2\right).
\end{aligned}
$$

(e)

$$\begin{aligned}
p^n &= \int d^3x T^{0n} \\
T^{0n} &= (\partial^0\phi)(\partial^n\phi) - g^{0n} \quad (anything) \\
&= (\partial^0\phi)(\partial^n\phi) \\
&= (\partial_0\phi)(\partial^n\phi) \\
p^n &= \int d^3x\dot{\phi}\partial^n\phi.
\end{aligned}$$

p^n is the momentum inside the field.

4. In part (a) of this question, the β should be as downstair index of A.

(a)

$$\mathscr{L} = -\frac{1}{4}(A_{\nu,\mu} - A_{\mu,\nu})(A^{\nu,\mu} - A^{\mu,\nu})$$

If both indices are space -like, i.e. α, $\beta = 1, 2, 3$, then the Lagrangian density is,

$$\mathscr{L} = -\frac{1}{2}(A_{\beta,\alpha} - A_{\alpha,\beta})^2 + \text{ other terms that don't contain } \alpha,\ \beta.$$

However, if one index is time-like, e.g. $\alpha = 0$, $\beta = 1, 2, 3$

$$\begin{aligned}
\mathscr{L} &= -\frac{1}{4}(A_{\beta,\alpha} - A_{\alpha,\beta})(A^{\beta,\alpha} - A^{\alpha,\beta}) - \frac{1}{4}(A_{\alpha,\beta} - A_{\beta,\alpha})(A^{\alpha,\beta} - A^{\beta,\alpha}) + \text{others} \\
&= \frac{1}{2}(A_{\beta,\alpha} - A_{\alpha,\beta})^2 + \text{others.}
\end{aligned}$$

Whatever the case may be, we obtain,

$$\Pi^{\alpha\beta} = \frac{\mathscr{L}}{\partial(\partial_\alpha A_\beta)} = \frac{\partial\mathscr{L}}{\partial(A_{\beta,\alpha})} = \mp(A_{\beta,\Pi^{\alpha\beta}=\alpha} - A_{\alpha,\beta}) = \mp F_{\alpha\beta}$$

while the "-" sign means that both α and β are space-like and the "+" sign means that one of these is time-like. Combining these options we obtain,

$$\Pi^{\alpha\beta} = \mp F_{\alpha\beta} = -F^{\alpha\beta}.$$

(b) For a scalar field ϕ we have,

$$T^{\mu\nu} = \Pi^\mu\partial^\nu\phi - g^{\mu\nu}\mathscr{L},$$

Whereas for a multi-component vector field A we can write,

$$\begin{aligned} T^{\mu\nu} &= \Pi^{\mu\beta}\partial^\nu A_\beta - g^{\mu\nu}\mathscr{L} \\ &= -F^{\mu\beta}\partial^\nu A_\beta - g^{\mu\nu}\left(-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}\right) \\ &= -F^{\mu\beta}\partial^\nu A_\beta + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}. \end{aligned}$$

(c)

$$T^{\nu\mu} = -F^{\nu\beta}\partial^\mu A_\beta + \frac{1}{4}g^{\nu\mu}F_{\alpha\beta}F^{\alpha\beta} \neq T^{\mu\nu}.$$

This is clearly not symmetric.

(d) We need to symmetrize. Add $\partial_\lambda X^{\lambda\mu\nu}$ where $X^{\lambda\mu\nu} = F^{\mu\lambda}A^\nu$, and also show that $X^{\lambda\mu\nu} = -X^{\mu\lambda\nu}$.

$$\begin{aligned} \text{Since} \qquad X^{\lambda\mu\nu} &= F^{\mu\lambda}A^\nu, \qquad \text{we obtain,} \\ X^{\mu\lambda\nu} &= F^{\lambda\mu}A^\nu = -F^{\mu\lambda}A^\nu = -X^{\lambda\mu\nu} \qquad \text{since } F \text{ is antisymmetric,} \end{aligned}$$

adding the divergence of X to T leads to,

$$\begin{aligned} \bar{T}^{\mu\nu} &= -F^{\mu\beta}\partial^\nu A_\beta + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \partial_\lambda X^{\lambda\mu\nu} \\ &= -F^{\mu\beta}\partial^\nu A_\beta + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \partial_\lambda(F^{\mu\lambda}A^\nu) \\ &= -F^{\mu\beta}\partial^\nu A_\beta + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + (\partial_\lambda F^{\mu\lambda})A^\nu + F^{\mu\lambda}\partial_\lambda A^\nu \\ &= -F^{\mu\beta}\partial^\nu A_\beta + F^{\mu\lambda}\partial_\lambda A^\nu + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \quad \text{since } \partial_\lambda F^{\mu\lambda} = 0 \text{ because of Maxwell.} \\ &= -F^{\mu\beta}\partial^\nu A_\beta + F^{\mu\beta}\partial_\beta A^\nu + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \\ &= F^{\mu\beta}(\partial_\beta A^\nu - \partial^\nu A_\beta) + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \\ \text{Now} \qquad F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ F_{\beta\nu} &= \partial_\beta A_\nu - \partial_\nu A_\beta \\ \Rightarrow \quad F_\beta{}^\nu &= \partial_\beta A^\nu - \partial^\nu A_\beta \\ \text{Hence} \quad \bar{T}^{\mu\nu} &= F^{\mu\beta}F_\beta{}^\nu + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}, \end{aligned}$$

and interchanging μ and ν yields,

$$\bar{T}^{\nu\mu} = F^{\nu\beta}F_\beta{}^\mu + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \qquad (\therefore \quad g^{\mu\nu} = g^{\nu\mu}).$$

We now show that $\bar{T}$ is indeed symmetric. The second term on the R.H.S is symmetric. We only need to verify symmetry for the first term.

$$\begin{aligned}
F^{\mu 0}F_0{}^{\nu} + F^{\mu 1}F_1{}^{\nu} + F^{\mu 2}F_2{}^{\nu} + F^{\mu 3}F_3{}^{\nu} &= \bar{T}^{\mu\nu} \quad \text{(first term only)} \\
F^{\nu 0}F_0{}^{\mu} + F^{\nu 1}F_1{}^{\mu} + F^{\nu 2}F_2{}^{\mu} + F^{\nu 3}F_3{}^{\mu} &= \bar{T}^{\nu\mu} \quad \text{(first term only)} \\
\text{Now} \quad F^{\mu 0} = F_0{}^{\mu} \quad &\text{and} \quad F^{\mu n} = -F_n{}^{\mu} \\
\text{while} \quad F_0{}^{\nu} = F^{\nu 0} \quad &\text{and} \quad F_n{}^{\nu} = -F^{\nu n}
\end{aligned}$$

which results in each term in $-\bar{T}^{\mu\nu}$ being equal to corresponding term in $\bar{T}^{\nu\mu}$.

$$\text{Hence} \quad \bar{T}^{\mu\nu} = \bar{T}^{\nu\mu}.$$

By adding $(\partial_\lambda X^{\lambda\mu\nu})$, $\bar{T}^{\nu\mu}$ becomes symmetric. We like to have symmetric tensors, especially in general relativity.

(e)

$$\bar{T}^{00} = F^{0\beta}F_\beta{}^0 + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}$$

Now

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$$F_\mu{}^\nu = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ E_1 & 0 & -B_3 & B_2 \\ E_2 & B_3 & 0 & -_1 \\ E_3 & -B_2 & B_1 & 0 \end{pmatrix}$$

$$\begin{aligned}
\therefore \quad F^{0\beta}F_\beta{}^0 &= +|E|^2 \\
\bar{T}^{00} &= |E|^2 - \frac{1}{2}\left(|E|^2 - |B|^2\right) = \frac{1}{2}\left(|E|^2 + |B|^2\right)
\end{aligned}$$

which is the energy density.

(f)

$$\begin{aligned}
\bar{T}^{\mu\nu} &= F^{\mu\beta}F_{\beta}{}^{\nu} + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \\
\bar{T}^{0i} &= F^{0\beta}F_{\beta}{}^{i} + \frac{1}{4}g^{0i}F_{\alpha\beta}F^{\alpha\beta} \\
g^{0i} &= 0 \\
\therefore \quad \bar{T}^{0i} &= F^{0\beta}F_{\beta}{}^{i} \\
&= F^{00}F_{0}{}^{i} + F^{0n}F_{n}{}^{i} \\
&= F^{0n}F_{n}{}^{i} = F^{0j}F_{j}{}^{i}
\end{aligned}$$

Now

$$\begin{aligned}
\bar{T}^{0i} &= F^{0j}F_{j}{}^{i} = \epsilon^{ijk}E_jB_k = (\vec{E}\times\vec{B})_i \\
\therefore \quad \bar{T}^{0i} &= (\vec{E}\times\vec{B})_i \\
P^i &= \int d^3x\bar{T}^{0i} = \int d^3x(\vec{E}\times\vec{B})_i.
\end{aligned}$$

$(\vec{E}\times\vec{B})_i$ is the Poynting vector or energy flow per unit volume in the i direction.

(g) We need to show $\partial_\mu\bar{T}^{0\mu} = 0$ i.e. $\partial_t\bar{T}^{00} + \nabla\cdot\bar{\mathbf{T}}^{\mathbf{0}} = 0$, where $\bar{\mathbf{T}}^{0} = (\bar{T}^{0i}) = (\bar{T}^{01}, \bar{T}^{02}, \bar{T}^{03})$.

Now
$$\begin{aligned}
\partial_t\bar{T}^{00} &= \partial_t\,\frac{1}{2}\big(\mathbf{E}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{B}\big) \quad \text{from the previous parts} \\
&= \mathbf{E}\cdot\dot{\mathbf{E}}+\mathbf{B}\cdot\dot{\mathbf{B}} \\
&= \mathbf{E}\cdot\nabla\times\mathbf{B}-\mathbf{B}\cdot\nabla\times\mathbf{E} \quad \text{Using Maxwell's equations in source-free environmen} \\
&= -\nabla\cdot\big(\mathbf{E}\times\mathbf{B}\big) \\
&= -\partial_i\bar{\mathbf{T}}^{0i}.
\end{aligned}$$

Hence $\partial_\mu\bar{T}^{0\mu} = 0$ as desired, i.e. $\bar{T}^{0}$ forms energy tensor, whose time-like term is the energy density and space-like term shows the flux of energy.