Solution Assignment 4: Quantum Field Theory

1. The Fourier decomposite of $\hat{x}_j$ leads to

$$\hat{x}_j \;=\; \frac{1}{\sqrt{N}}\sum_{\mathbf{k}} \hat{x}_k e^{ikja}$$

While the creation annihilation operators are given by,

$$\hat{x}_k \;=\; \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_{\mathbf{k}} + \hat{a}^{\dagger}_{-\mathbf{k}}).$$

The equation above yields,

$$\begin{aligned}\hat{x}_j \;&=\; \frac{1}{\sqrt{N}}\sqrt{\frac{\hbar}{2m\omega}}\sum_{\mathbf{k}}\Big(\hat{a}_{\mathbf{k}}e^{ikja} + \hat{a}^{\dagger}_{-\mathbf{k}}e^{ikja}\Big)\\ &=\; \sqrt{\frac{\hbar}{2m\omega N}}\sum_{\mathbf{k}}\Big(\hat{a}_{\mathbf{k}}e^{ikja} + \hat{a}^{\dagger}_{+\mathbf{k}}e^{-ikja}\Big),\end{aligned}$$

where we have replaced $\mathbf{k}$ by $-\mathbf{k}$ in the second term. This is allowed because the sum is over $\mathbf{k}$, which symmetrically runs from -ve to +ve values. We will call this the "mode expansion formula" in our course.

2. Heisenberg's uncertainty relationship for position and momentum is:

$$\begin{aligned}[\hat{x},\hat{p}] \;&=\; i\hbar\\ \text{Now}\quad \hat{a}_{\mathbf{k}} \;&=\; \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_{\mathbf{k}} + \frac{i}{m\omega}\hat{p}_{\mathbf{k}}\right)\\ \hat{a}^{\dagger}_{\mathbf{k}} \;&=\; \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x}_{\mathbf{k}} - \frac{i}{m\omega}\hat{p}_{\mathbf{k}}\right)\\ \therefore\quad [\hat{a}_{\mathbf{k}},\hat{a}^{\dagger}_{\mathbf{k}}] \;&=\; \left(\frac{m\omega}{2\hbar}\right)\left[\hat{x}_k + \frac{i}{m\omega}\hat{p}_k, \hat{x}_k - \frac{i}{m\omega}\hat{p}_k\right]\\ &=\; \left(\frac{m\omega}{2\hbar}\right)\left(-\frac{i}{m\omega}[\hat{x}_k,\hat{p}_k] + \frac{i}{m\omega}[\hat{p}_k,\hat{x}_k]\right)\\ &=\; \left(\frac{1}{2\hbar}\right)\Big(-i(i\hbar) + i(-i\hbar)\Big)\\ &=\; \left(\frac{1}{2\hbar}\right)(2\hbar) = 1.\end{aligned}$$

3. (a) Find the overlap between the two -particle states:

$$\begin{aligned}\langle \mathbf{p}'\ \mathbf{q}'|\mathbf{p}\ \mathbf{q}\rangle \;&=\; \langle \mathbf{p}'\ \mathbf{q}'|\hat{a}^{\dagger}_p, \hat{a}^{\dagger}_q|0\rangle\\ &=\; \langle 0|\hat{a}_{p'}, \hat{a}_{q'}\ \hat{a}^{\dagger}_p, \hat{a}^{\dagger}_q|0\rangle\\ \text{Now}\qquad \hat{a}_q\ \hat{a}^{\dagger}_p \;&=\; \delta^{(3)}_{\mathbf{p},\mathbf{q}} \pm\ \hat{a}^{\dagger}_p\ \hat{a}_{q'},\end{aligned}$$

where " + " is for bosons and " − " for fermions.

$$\begin{aligned}
&= \langle 0|\hat{a}_{p'}\Big(\delta^{(3)}(\mathbf{q}'-\mathbf{p}) \pm \hat{a}^\dagger_p\ \hat{a}_{q'}\Big)\hat{a}^\dagger_q|0\rangle \\
&= \langle 0|\Big(\hat{a}_{p'}\delta^{(3)}(\mathbf{q}'-\mathbf{p}) \pm \hat{a}_{p'}\hat{a}^\dagger_p\ \hat{a}_{q'}\Big)\hat{a}^\dagger_q|0\rangle \\
&= \langle 0|\Big(\hat{a}_{p'}\hat{a}^\dagger_q\delta^{(3)}(\mathbf{q}'-\mathbf{p}) \pm \hat{a}_{p'}\hat{a}^\dagger_p\ \hat{a}_{q'}\hat{a}^\dagger_q\Big)|0\rangle \\
&= \delta^{(3)}(\mathbf{p}'-\mathbf{q})\delta^{(3)}(\mathbf{q}'-\mathbf{p}) \pm \langle 0|\hat{a}_{p'}\hat{a}^\dagger_p\ \hat{a}_{q'}\hat{a}^\dagger_q|0\rangle
\end{aligned}$$

Look at the second term:

$$\begin{aligned}
&= \langle 0\Big(\delta^{(3)}(\mathbf{p}'-\mathbf{p}) \pm \hat{a}^\dagger_p\hat{a}_{p'}\Big)\Big(\delta^{(3)}(\mathbf{q}'-\mathbf{q}) \pm \hat{a}^\dagger_q\hat{a}_{q'}\Big)|0\rangle \\
&= \langle 0\Big(\delta^{(3)}(\mathbf{p}'-\mathbf{p})\delta^{(3)}(\mathbf{q}'-\mathbf{q}) \pm \delta^{(3)}(\mathbf{p}'-\mathbf{p})\hat{a}^\dagger_q\hat{a}_{q'} \pm \delta^{(3)}(\mathbf{q}'-\mathbf{q})\hat{a}^\dagger_p\hat{a}_{p'} + \hat{a}^\dagger_p\hat{a}_{p'}\ \hat{a}^\dagger_q\hat{a}_{q'}|0\rangle
\end{aligned}$$

All but the first term are zero.

$$\langle \mathbf{p}'\ \mathbf{q}'|\mathbf{p}\ \mathbf{q}\rangle = \delta^{(3)}(\mathbf{p}'-\mathbf{q})\delta^{(3)}(\mathbf{q}'-\mathbf{p}) \pm \delta^{(3)}(\mathbf{p}'-\mathbf{p})\delta^{(3)}(\mathbf{q}'-\mathbf{q}).$$

(b)

$$\begin{aligned}
\langle \mathbf{x}\ \mathbf{y}\ |\ \mathbf{p}\ \mathbf{q}\rangle &= \frac{1}{\sqrt{2}}\int d^3p'\ d^3q'\ \langle \mathbf{x}\ \mathbf{y}\ |\ \mathbf{p}'\ \mathbf{q}'\rangle\langle \mathbf{p}'\ \mathbf{q}'|\mathbf{p}\ \mathbf{q}\rangle \\
\text{Let}\quad \langle \mathbf{x}\ |\ \mathbf{p}\rangle &= \phi_{\mathbf{p}'}(\mathbf{x}) \\
\therefore\quad \langle \mathbf{x}\ \mathbf{y}|\mathbf{p}\ \mathbf{q}\rangle &= \frac{1}{\sqrt{2}}\int d^3p'\ d^3q'\ \phi_{\mathbf{p}'}(\mathbf{x})\ \phi_{\mathbf{q}'}(\mathbf{y})\langle \mathbf{p}'\ \mathbf{q}'|\mathbf{p}\ \mathbf{q}\rangle \\
\text{Now}\quad \langle \mathbf{p}'\ \mathbf{q}'|\mathbf{p}\ \mathbf{q}\rangle &= \delta^{(3)}(\mathbf{p}'-\mathbf{q})\delta^{(3)}(\mathbf{q}'-\mathbf{p}) \pm \delta^{(3)}(\mathbf{p}'-\mathbf{p})\delta^{(3)}(\mathbf{q}'-\mathbf{q}) \\
\therefore\quad \langle \mathbf{x}\ \mathbf{y}|\mathbf{p}\ \mathbf{q}\rangle &= \frac{1}{\sqrt{2}}\int d^3p'\ d^3q'\ \phi_{\mathbf{p}'}(\mathbf{x})\ \phi_{\mathbf{q}'}(\mathbf{y})\Big(\delta^{(3)}(\mathbf{p}'-\mathbf{q})\delta^{(3)}(\mathbf{q}'-\mathbf{p}) \\
&\pm\ \delta^{(3)}(\mathbf{p}'-\mathbf{p})\delta^{(3)}(\mathbf{q}'-\mathbf{q})\Big) \\
&= \frac{1}{\sqrt{2}}\Big(\phi_{\mathbf{q}}(\mathbf{x})\ \phi_{\mathbf{p}}(\mathbf{y}) \pm \phi_{\mathbf{p}}(\mathbf{x})\ \phi_{\mathbf{q}}(\mathbf{y})\Big).
\end{aligned}$$

4. Please do this yourself.

5. Let's first calculate, out of our desire, the commutators of the b_i's. We are provided with the commutator relations for the a's,

$$
\begin{aligned}
[\hat{a}_i, \hat{a}_j^\dagger] &= \delta_{i,j} \\
\big[\hat{b}_0, \hat{b}_0^\dagger\big] &= [\hat{a}_3, \hat{a}_3^\dagger] = 1 \\
\big[\hat{b}_1, \hat{b}_1^\dagger\big] &= \left[-\frac{1}{\sqrt{2}}(\hat{a}_1 - i\hat{a}_2),\ -\frac{1}{\sqrt{2}}(\hat{a}_1^\dagger + i\hat{a}_2^\dagger)\right] \\
&= \frac{1}{2}\big[(\hat{a}_1 - i\hat{a}_2), (\hat{a}_1^\dagger + i\hat{a}_2^\dagger)\big] \\
&= \frac{1}{2}\bigg(\big[\hat{a}_1, \hat{a}_1^\dagger\big] + \big[\hat{a}_2, \hat{a}_2^\dagger\big]\bigg) = 1. \\
\big[\hat{b}_1, \hat{b}_2^\dagger\big] &= \left[-\frac{1}{\sqrt{2}}(\hat{a}_1 - i\hat{a}_2),\ -\frac{1}{\sqrt{2}}(\hat{a}_1^\dagger - i\hat{a}_2^\dagger)\right] \\
&= -\frac{1}{2}\bigg(\big[\hat{a}_1, \hat{a}_1^\dagger\big] - \big[\hat{a}_2, \hat{a}_2^\dagger\big]\bigg) = 0 \\
&\text{etc.}
\end{aligned}
$$

We now like to express $\hat{a}_i^\dagger \hat{a}_i$ in terms of $\hat{b}_i^\dagger \hat{b}_i$. For this purpose calculate the following.

$$
\begin{aligned}
\text{Now} \quad \hat{b}_1^\dagger + \hat{b}_2^\dagger &= \frac{1}{\sqrt{2}}(\hat{a}_1^\dagger - \hat{a}_1^\dagger - i\hat{a}_2^\dagger - i\hat{a}_2^\dagger) = -i\sqrt{2}(\hat{a}_2^\dagger) \\
\hat{a}_2^\dagger &= \frac{i}{\sqrt{2}}(\hat{b}_1^\dagger + \hat{b}_2^\dagger) \\
\text{and} \quad \hat{a}_2 &= -\frac{i}{\sqrt{2}}(\hat{b}_1 + \hat{b}_2). \\
\text{Similarly} \quad \hat{b}_2^\dagger - \hat{b}_1^\dagger &= \sqrt{2}(\hat{a}_1^\dagger) \\
\hat{a}_1^\dagger &= \frac{1}{\sqrt{2}}(\hat{b}_2^\dagger - \hat{b}_1^\dagger) \\
\text{and} \quad \hat{a}_1 &= \frac{1}{\sqrt{2}}(\hat{b}_2 - \hat{b}_1). \\
H &= \hbar\omega \sum_i \left(\hat{a}_i^\dagger \hat{a}_i + \frac{1}{2}\right) \\
&= \hbar\omega\big(\hat{a}_1^\dagger \hat{a}_1 + \hat{a}_2^\dagger \hat{a}_2 + \hat{a}_3^\dagger \hat{a}_3 + \frac{1}{2}\big) \\
&= \hbar\omega\left(\frac{1}{2}(\hat{b}_2^\dagger - \hat{b}_1^\dagger)(\hat{b}_2 - \hat{b}_1) + \frac{i}{2}(\hat{b}_1^\dagger + \hat{b}_2^\dagger)(-i)(\hat{b}_1 + \hat{b}_2) + \hat{b}_0^\dagger \hat{b}_0 + \frac{1}{2}\right) \\
&= \hbar\omega\left(\frac{1}{2}(\hat{b}_2^\dagger \hat{b}_2 - \hat{b}_2^\dagger \hat{b}_1 - \hat{b}_1^\dagger \hat{b}_2 + \hat{b}_1^\dagger \hat{b}_1) + \frac{1}{2}(\hat{b}_1^\dagger \hat{b}_1 + \hat{b}_1^\dagger \hat{b}_2 + \hat{b}_2^\dagger \hat{b}_1 + \hat{b}_2^\dagger \hat{b}_2) + \hat{b}_0^\dagger \hat{b}_0 + \frac{1}{2}\right) \\
&= \hbar\omega\left(\hat{b}_0^\dagger \hat{b}_0 + \hat{b}_2^\dagger \hat{b}_2 + \hat{b}_1^\dagger \hat{b}_1 + \frac{1}{2}\right) \\
&= \hbar\omega \sum_i \left(\hat{b}_i^\dagger \hat{b}_i + \frac{1}{2}\right).
\end{aligned}
$$

$$\text{So} \qquad H \;=\; \hbar\omega \sum_{i=0,1,2}\left(\hat{a}_i^\dagger \hat{a}_i + \frac{1}{2}\right) = \hbar\omega \sum_{i=0,1,2}\left(\hat{b}_i^\dagger \hat{b}_i + \frac{1}{2}\right).$$

6. The anticommutator for the fermionic field operators can be found in the following fashion.

$$\begin{aligned}
\left\{\hat{\psi}(\mathbf{x}),\ \psi^\dagger\hat{(}\mathbf{y})\right\} &= \left\{\frac{1}{\sqrt{V}}\sum_{\mathbf{q}} e^{+i\mathbf{q}.\mathbf{x}}\hat{c}_{\mathbf{q}},\ \frac{1}{\sqrt{V}}\sum_{\mathbf{p}} e^{-i\mathbf{p}.\mathbf{y}}\hat{c}_{\mathbf{p}}^\dagger\right\} \\
&= \frac{1}{V}\sum_{\mathbf{p},\mathbf{q}} e^{+i\mathbf{q}.\mathbf{x}}\ e^{-i\mathbf{p}.\mathbf{y}}\left\{\hat{c}_{\mathbf{q}},\ \hat{c}_{\mathbf{p}}^\dagger\right\} \\
&= \frac{1}{V}\sum_{\mathbf{p},\mathbf{q}} e^{+i\mathbf{q}.\mathbf{x}}\ e^{-i\mathbf{p}.\mathbf{y}}\delta_{\mathbf{q},\mathbf{p}} \\
&= \frac{1}{V}\sum_{\mathbf{p}} e^{i\mathbf{q}.(\mathbf{x}-\mathbf{y})} \\
&= \delta^{(3)}(\mathbf{x}-\mathbf{y}).
\end{aligned}$$