Solution Assignment 5: Quantum Field Theory

1. The second-quantization version of a single particle operator is:

$$\hat{A} = \sum_{\alpha,\,\beta} \langle \alpha | \hat{A} | \beta \rangle \hat{a}^{\dagger}_{\alpha} \hat{a}_{\beta}.$$

If $|\alpha\rangle$ and $|\beta\rangle$ are momentum eigenstates, show that this operator in the position space is given by

$$\hat{A} = \int d^3x \; A(\mathbf{x}) \; \hat{\psi}^{\dagger}(\mathbf{x}) \; \hat{\psi}(\mathbf{x}).$$

I am looking for a neat and systematic solution.

Answer 1

We start off by calculating,

$$\begin{aligned} \langle \alpha | \hat{A} | \beta \rangle &= \int\int d^3x \; d^3y \langle \alpha | \mathbf{x} \rangle \langle \mathbf{x} | A | \mathbf{y} \rangle \langle \mathbf{y} | \beta \rangle \\ &= \int\int d^3x \; d^3y \; \phi^*_{\alpha}(\mathbf{x}) \phi_{\beta}(\mathbf{y}) \langle \mathbf{x} | A | \mathbf{y} \rangle \end{aligned}$$

where $\langle \mathbf{y} | \beta \rangle = \phi_{\beta}(\mathbf{y})$ etc.

Therefore

$$\begin{aligned} \hat{F} &= \sum_{\alpha,\,\beta} \langle \alpha | \hat{A} | \beta \rangle \hat{a}^{\dagger}_{\alpha} \hat{a}_{\beta} \\ &= \sum_{\alpha,\,\beta} \int\int d^3x \; d^3y \; \phi^*_{\alpha}(\mathbf{x}) \phi_{\beta}(\mathbf{y}) \langle \mathbf{x} | \hat{A} | \mathbf{y} \rangle \hat{a}^{\dagger}_{\alpha} \hat{a}_{\beta} \end{aligned}$$

Now

$$\begin{aligned} \phi_{\beta}(\mathbf{y}) &= \frac{1}{\sqrt{V}} e^{+i\mathbf{p}_{\beta}\cdot\mathbf{y}} \\ \phi^*_{\alpha}(\mathbf{x}) &= \frac{1}{\sqrt{V}} e^{-i\mathbf{p}_{\alpha}\cdot\mathbf{x}} \\ \hat{F} &= \frac{1}{V} \sum_{\alpha,\,\beta} \int_x \int_y d^3x \; d^3y \; e^{-i\mathbf{p}_{\alpha}\cdot\mathbf{x}} \; e^{+i\mathbf{p}_{\beta}\cdot\mathbf{y}} \langle \mathbf{x} | \hat{A} | \mathbf{y} \rangle \hat{a}^{\dagger}_{\alpha} \hat{a}_{\beta} \end{aligned}$$

Since

$$\psi^{\dagger}(\mathbf{x}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}_{\alpha}} e^{-i\mathbf{p}_{\alpha}\cdot\mathbf{x}} \; \hat{a}^{\dagger}_{\alpha} \quad \text{and} \quad \psi(\mathbf{y}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}_{\beta}} e^{-i\mathbf{p}_{\beta}\cdot\mathbf{x}} \; \hat{a}_{\beta}$$

$$\hat{F} = \int_x \int_y d^3x \; d^3y \; \langle \mathbf{x} | \hat{A} | \mathbf{y} \rangle \; \hat{\psi}^{\dagger}(\mathbf{x}) \; \hat{\psi}(\mathbf{y})$$

If $\hat{A}$ is a single particle operator,

$$\begin{aligned}
\langle \mathbf{x}|\hat{A}|\mathbf{y}\rangle &= A(\mathbf{x})\langle \mathbf{x}|\mathbf{y}\rangle = A(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y}) \\
&= \int_x\int_y d^3x d^3y Q A(\mathbf{x})\delta^{(3)}(\mathbf{x}-\mathbf{y})\hat{\psi}^\dagger(\mathbf{x})\ \hat{\psi}(\mathbf{y}) \\
\hat{F} &= \int d^3x\ A(\mathbf{x})\ \hat{\psi}^\dagger(\mathbf{x})\ \hat{\psi}(\mathbf{x}) \\
&= \int d^3x\ \hat{\psi}^\dagger(\mathbf{x})\ A(\mathbf{x})\ \hat{\psi}(\mathbf{x}) \quad \text{as desired.}
\end{aligned}$$

2. (a) Determine $\{\hat{\psi}(\mathbf{x}),\, \hat{c}^\dagger_{\mathbf{p}}\}$.

   (b) Consider the fermionic particles in states $\mathbf{p}$ and $\mathbf{q}$ generated by the operators,

$$\hat{c}^\dagger_{\mathbf{q}}\hat{c}^\dagger_{\mathbf{p}}|\,0\rangle = |\,\mathbf{pq}\rangle.$$

   Find the representation of $|\,\mathbf{pq}\rangle$ in the position basis. For this purpose, you will calculate $\langle \mathbf{x}_1\mathbf{x}_2|\,\mathbf{pq}\rangle$ while using the (anti) commutator result derived in part (a). [Hint: $|x\rangle = \hat{\psi}^\dagger(\mathbf{x})|\,0\rangle$].

Answer 2

(a)

$$\begin{aligned}
\{\hat{\psi}(\mathbf{x}),\, \hat{c}^\dagger_{\mathbf{p}}\} &= \left\{\frac{1}{\sqrt{V}}\sum_{\mathbf{q}} e^{+i\mathbf{q}\cdot\mathbf{x}}\ \hat{c}_{\mathbf{q}}\,,\ \hat{c}^\dagger_{\mathbf{p}}\right\} \\
\text{say}\quad \frac{e^{+i\mathbf{q}\cdot\mathbf{x}}}{\sqrt{V}} &= \phi_q(\mathbf{x}) = \langle \mathbf{x}|\mathbf{q}\rangle
\end{aligned}$$

The anticommutator is,

$$\begin{aligned}
\sum_{\mathbf{q}}\{\phi_q(\mathbf{x})\hat{c}_{\mathbf{q}}\,,\ \hat{c}^\dagger_{\mathbf{p}}\} &= \sum_{\mathbf{q}}\phi_q(\mathbf{x})\{\hat{c}_{\mathbf{q}}\,,\ \hat{c}^\dagger_{\mathbf{p}}\} \\
&= \sum_{\mathbf{q}}\phi_q(\mathbf{x})\ \delta_{\mathbf{q},\mathbf{P}} = \phi_p(\mathbf{x}) = \langle \mathbf{x}|\mathbf{p}\rangle.
\end{aligned}$$

We can also do some additional ramblings as here.

$$\begin{aligned}
\therefore \quad \left\{\hat{\psi}(\mathbf{x}),\, \hat{c}^\dagger_{\mathbf{p}}\right\} &= \langle \mathbf{x}|\mathbf{p}\rangle \\
\Rightarrow \quad \hat{\psi}(\mathbf{x})\hat{c}^\dagger_{\mathbf{p}} + \hat{c}^\dagger_{\mathbf{p}}\hat{\psi}(\mathbf{x}) &= \langle \mathbf{x}|\mathbf{p}\rangle \\
\hat{\psi}(\mathbf{x})\hat{c}^\dagger_{\mathbf{p}} &= -\hat{c}^\dagger_{\mathbf{p}}\hat{\psi}(\mathbf{x}) + \langle \mathbf{x}|\mathbf{p}\rangle.
\end{aligned}$$

(b) We have to determine $\langle \mathbf{x}_1, \mathbf{x}_2|\mathbf{p}, \mathbf{q}\rangle$, where $|\mathbf{pq}\rangle = \hat{c}^\dagger_{\mathbf{q}}\hat{c}^\dagger_{\mathbf{p}}|\,0\rangle$. Note that $\langle \mathbf{x}_1, \mathbf{x}_2| = \langle 0|\hat{\psi}(\mathbf{x}_1)\hat{\psi}(\mathbf{x}_2)$.

Let's first determine $\langle 0|\hat{\psi}(\mathbf{x}_1)\hat{\psi}(\mathbf{x}_2)\hat{c}^\dagger_{\mathbf{q}}\hat{c}^\dagger_{\mathbf{p}}|\,0\rangle$. We like to take all the annihilation operators at the right side. From the previous part, we have,

$$\begin{aligned}
\hat{\psi}(\mathbf{x}_1)\hat{\psi}(\mathbf{x}_2)\hat{c}^\dagger_{\mathbf{q}}\hat{c}^\dagger_{\mathbf{p}} &= \hat{\psi}(\mathbf{x}_1)\Big[-\hat{c}^\dagger_{\mathbf{q}}\hat{\psi}(\mathbf{x}_2) + \langle \mathbf{x}_2|\mathbf{q}\rangle\Big]\hat{c}^\dagger_{\mathbf{p}} \\
&= -\hat{\psi}(\mathbf{x}_1)\hat{c}^\dagger_{\mathbf{q}}\hat{\psi}(\mathbf{x}_2)\hat{c}^\dagger_{\mathbf{p}} + \langle \mathbf{x}_2|\mathbf{q}\rangle\hat{\psi}(\mathbf{x}_1)\hat{c}^\dagger_{\mathbf{p}} \\
&= -\hat{\psi}(\mathbf{x}_1)\hat{c}^\dagger_{\mathbf{q}}\Big(-\hat{c}^\dagger_{\mathbf{p}}\hat{\psi}(\mathbf{x}_2) + \langle \mathbf{x}_2|\mathbf{p}\rangle\Big) + \langle \mathbf{x}_2|\mathbf{q}\rangle\Big(-\hat{c}^\dagger_{\mathbf{p}}\hat{\psi}(\mathbf{x}_1) \\
&\quad + \langle \mathbf{x}_1|\mathbf{p}\rangle\Big) \\
&= \hat{\psi}(\mathbf{x}_1)\hat{c}^\dagger_{\mathbf{q}}\hat{c}^\dagger_{\mathbf{p}}\hat{\psi}(\mathbf{x}_2) - \langle \mathbf{x}_2|\mathbf{p}\rangle\hat{\psi}(\mathbf{x}_1)\hat{c}^\dagger_{\mathbf{q}} - \langle \mathbf{x}_2|\mathbf{q}\rangle\hat{c}^\dagger_{\mathbf{p}}\hat{\psi}(\mathbf{x}_1) \\
&\quad + \langle \mathbf{x}_2|\mathbf{q}\rangle\langle \mathbf{x}_1|\mathbf{p}\rangle
\end{aligned}$$

The third term acting on $|\,0\rangle$ does not lead us anywhere, we don't need to consider it. So,

$$\begin{aligned}
&\Big(-\hat{c}^\dagger_{\mathbf{q}}\hat{\psi}(\mathbf{x}_1) + \langle \mathbf{x}_1|\mathbf{q}\rangle\Big)\hat{c}^\dagger_{\mathbf{p}}\hat{\psi}(\mathbf{x}_2) - \langle \mathbf{x}_2|\mathbf{p}\rangle\Big(-\hat{c}^\dagger_{\mathbf{q}}\hat{\psi}(\mathbf{x}_1) + \langle \mathbf{x}_1|\mathbf{q}\rangle\Big) + \langle \mathbf{x}_2|\mathbf{q}\rangle\langle \mathbf{x}_1|\mathbf{p}\rangle \\
&= -\hat{c}^\dagger_{\mathbf{q}}\hat{\psi}(\mathbf{x}_1)\hat{c}^\dagger_{\mathbf{p}}\hat{\psi}(\mathbf{x}_2) + \langle \mathbf{x}_1|\mathbf{q}\rangle\hat{c}^\dagger_{\mathbf{p}}\hat{\psi}(\mathbf{x}_2) + \langle \mathbf{x}_2|\mathbf{p}\rangle\hat{c}^\dagger_{\mathbf{q}}\hat{\psi}(\mathbf{x}_1) - \langle \mathbf{x}_2|\mathbf{p}\rangle\langle \mathbf{x}_1|\mathbf{q}\rangle \\
&\quad + \langle \mathbf{x}_2|\mathbf{q}\rangle\langle \mathbf{x}_1|\mathbf{p}\rangle.
\end{aligned}$$

The second and third terms need not be proceeded any further, because when they act on $|0\rangle$ to the right, they disappear. Similarly the first term above yields,

$$
\begin{aligned}
&= -\hat{c}^\dagger_{\mathbf{q}}\left(\hat{c}^\dagger_{\mathbf{p}}\hat{\psi}(\mathbf{x}_1) + \langle \mathbf{x}_1|\mathbf{p}\rangle\right)\hat{\psi}(\mathbf{x}_2)\\
&= \hat{c}^\dagger_{\mathbf{q}}\hat{c}^\dagger_{\mathbf{p}}\hat{\psi}(\mathbf{x}_1)\hat{\psi}(\mathbf{x}_2) - \langle \mathbf{x}_1|\mathbf{p}\rangle\hat{c}^\dagger_{\mathbf{q}}\hat{\psi}(\mathbf{x}_2).
\end{aligned}
$$

These terms don't matter as well as they annihilate vacuum. Hence Finally,

$$
\begin{aligned}
\langle \mathbf{x}_1, \mathbf{x}_2|\mathbf{p}, \mathbf{q}\rangle &= \langle \mathbf{x}_2|\mathbf{q}\rangle\langle \mathbf{x}_1|\mathbf{p}\rangle - \langle \mathbf{x}_2|\mathbf{p}\rangle\langle \mathbf{x}_1|\mathbf{q}\rangle\\
&= \frac{1}{V}\left(e^{+i(\mathbf{q}.\mathbf{x}_2+\mathbf{p}.\mathbf{x}_1)} - e^{-i(\mathbf{p}.\mathbf{x}_2+\mathbf{q}.\mathbf{x}_1)}\right).
\end{aligned}
$$

This has the correct antisymmetric form. The antisymmetrization for the fermions is built into the anti-commutation relation for the field operators.

3. (a) Consider the interaction between two particles described by

$$V(\mathbf{x}) = A\delta^{(3)}(\mathbf{x}),$$

where A is a constant. Find the second-quantization form of the interaction.

(b) If the inter-particle interaction is described by the Yukawa potential

$$V(r) = \frac{Ae^{-\lambda r}}{r},$$

determine the second-quantization form of the interaction. Convince yourself that the coulomb interaction is the long range form of the Yukawa interaction, where $A = \dfrac{q^2}{2\pi\varepsilon_0}$.

Answer 3

(a)

$$
\begin{aligned}
\widetilde{V}_{\mathbf{q}} &= \frac{1}{\curlyvee}\int d^3x A\delta^{(3)}(\mathbf{x})e^{-i\mathbf{q}\cdot\mathbf{x}} \\
&= A \\
\hat{V} &= \frac{A}{2}\sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{q}} \hat{a}^\dagger_{\mathbf{p}_2+\mathbf{q}}\ \hat{a}^\dagger_{\mathbf{p}_1-\mathbf{q}}\ \hat{a}_{\mathbf{p}_2}\ \hat{a}_{\mathbf{p}_1}
\end{aligned}
$$

(b)

$$
\begin{aligned}
V(r) &= \frac{Ae^{-\lambda r}}{r} \\
\widetilde{V}_{\mathbf{q}} &= \frac{1}{\mathcal{V}}\int d^3x\ V(r)\ e^{-i\mathbf{q}\cdot\mathbf{x}} \\
&= \frac{A}{\mathcal{V}}\int d^3x\ \frac{Ae^{-\lambda r}}{r}\ e^{-i\mathbf{q}\cdot\mathbf{x}} \\
\text{Now}\qquad d^3x &= r^2\sin\theta\ d\theta\ d\phi\ dr \\
\mathbf{q}\cdot\mathbf{x} &= qr\ \cos\theta \\
\therefore\qquad \tilde{V}_{\mathbf{q}} &= \frac{A}{\mathcal{V}}\int_r\int_\theta\int_\phi r^2\ \sin\theta\ d\theta\ d\phi\ dr\ \frac{Ae^{-\lambda r}}{r}\ e^{-iqr\cos\theta} \\
&= \frac{A}{\mathcal{V}}\ 2\pi\ \int_{r=0}^{\infty} r\ dr\int_{\theta=0}^{\pi}\ d\theta\ \sin\theta\ e^{-\lambda r}e^{-iqr\cos\theta} \\
&= \frac{A}{\mathcal{V}}\ 2\pi\ \int_{r=0}^{\infty} e^{-\lambda r}r\ dr\int_{\theta=0}^{\pi}\ d\theta\ \sin\theta\ e^{-iqr\cos\theta} \\
&= \frac{A}{\mathcal{V}}\ 2\pi\ \int_{r=0}^{\infty} e^{-\lambda r}r\ dr\int_{1}^{-1}\ d(\cos\theta)\ e^{-iqr(\cos\theta)} \\
&= \frac{A}{\mathcal{V}}\ 2\pi\ \int_{r=0}^{\infty} e^{-\lambda r}r\ dr\frac{e^{-iqr(\cos\theta)}}{-iqr}\Bigg|_{\cos\theta=1}^{\cos\theta=-1} \\
&= \frac{A}{\mathcal{V}}\ 2\pi\ \int_{r=0}^{\infty} e^{-\lambda r}r\ dr\left(\frac{e^{+iqr}-e^{-iqr}}{-iq}\right) \\
&= \frac{4\pi A}{-\mathcal{V}q}\int_{r=0}^{\infty}\sin(qr)e^{-\lambda r}\ dr \\
&= \frac{-8\pi A\lambda}{\mathcal{V}(q^2+\lambda^2)^2}.
\end{aligned}
$$

4. Consider two fermions a_1 and a_2.

(a) Show that Bogoliubov transformation

$$\begin{aligned}\hat{c}_1 &= u\hat{a}_1 + v\hat{a}_2^\dagger \\ \hat{c}_2^\dagger &= -v\hat{a}_1 + u\hat{a}_2^\dagger,\end{aligned}$$

where u and v are real, preserves the canonical anticommutation relations if $u^2 + v^2 = 1$

(b) Use this result to show that the Hamiltonian

$$H = \varepsilon(\hat{a}_1^\dagger\, \hat{a}_1 - \hat{a}_2\hat{a}_2^\dagger) + \Delta(\hat{a}_1^\dagger\, \hat{a}_2^\dagger + \hat{a}_1\hat{a}_2),$$

can be diagonalized in the form

$$H = \sqrt{\varepsilon^2 + \Delta^2}(\hat{c}_1^\dagger\, \hat{c}_1 + \hat{c}_2^\dagger\, \hat{c}_2 - 1).$$

(c) What is the ground-state energy of this Hamiltonian?

(d) Write out the ground-state wavefunction in terms of the original operators $\hat{c}_1^\dagger$ and $\hat{c}_2^\dagger$ and their corresponding vacuum $|\,0\rangle$, *i.e.*, $(\hat{c}_{1,2}|\,0\rangle = 0)$.

Answer 4

(a)

$$\begin{aligned}\{\hat{c}_1, \hat{c}_1^\dagger\} &= \{u\hat{a}_1 + v\hat{a}_2^\dagger, u\hat{a}_1^\dagger + v\hat{a}_2\} \\ &= u^2\{\hat{a}_1, \hat{a}_1^\dagger\} + v^2\{\hat{a}_2^\dagger, \hat{a}_2\} \\ &= u^2 + v^2 = 1 \\ \{\hat{c}_2, \hat{c}_2^\dagger\} &= \{-v\hat{a}_1 + u\hat{a}_2^\dagger, -v\hat{a}_1^\dagger + u\hat{a}_2\} \\ &= v^2\{\hat{a}_1, \hat{a}_1^\dagger\} + u^2\{\hat{a}_2^\dagger, \hat{a}_2\} \\ &= u^2 + v^2 = 1\end{aligned}$$

$$\begin{aligned}
\{\hat{c}_1, \hat{c}_2^\dagger\} &= \{u\hat{a}_1 + v\hat{a}_2^\dagger, -v\hat{a}_1^\dagger + u\hat{a}_2\} \\
&= -uv\{\hat{a}_1, \hat{a}_1^\dagger\} + uv\{\hat{a}_2^\dagger, \hat{a}_2\} \\
&= -uv(1) + uv(1) = 0.
\end{aligned}$$

(b)

$$\begin{aligned}
\hat{c}_1 &= u\hat{a}_1 + v\hat{a}_2^\dagger \\
\hat{c}_2^\dagger &= -v\hat{a}_1 + u\hat{a}_2^\dagger \\
\begin{pmatrix} \hat{c}_1 \\ \hat{c}_2^\dagger \end{pmatrix} &= \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2^\dagger \end{pmatrix} \\
\text{Let} \quad M &= \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \\
M^{-1} &= \begin{pmatrix} u & -v \\ v & u \end{pmatrix} \\
\hat{a}_1 &= u\hat{c}_1 - v\hat{c}_2^\dagger \\
\hat{a}_2^\dagger &= v\hat{c}_1 + u\hat{c}_2^\dagger \\
\hat{a}_1^\dagger &= u\hat{c}_1^\dagger - v\hat{c}_2 \\
\hat{a}_2 &= v\hat{c}_1^\dagger + u\hat{c}_2 \\
\mathcal{H} &= \varepsilon(\hat{a}_1^\dagger\hat{a}_1 - \hat{a}_2\hat{a}_2^\dagger) + \Delta(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2\hat{a}_2^\dagger) \\
&= \varepsilon\Big((u\hat{c}_1^\dagger - v\hat{c}_2)(u\hat{c}_1 - v\hat{c}_2^\dagger) - (v\hat{c}_1^\dagger + u\hat{c}_2)(v\hat{c}_1 + u\hat{c}_2^\dagger)\Big) \\
&+ \Delta\Big((u\hat{c}_1^\dagger - v\hat{c}_2)(v\hat{c}_1 + u\hat{c}_2^\dagger) + (v\hat{c}_1^\dagger + u\hat{c}_2)(u\hat{c}_1 - v\hat{c}_2^\dagger)\Big) \\
&= \varepsilon\Big(u^2\hat{c}_1^\dagger\hat{c}_1 - uv\hat{c}_1^\dagger\hat{c}_2^\dagger - vu\hat{c}_2\hat{c}_1 + v^2\hat{c}_2\hat{c}_2^\dagger - v^2\hat{c}_1^\dagger\hat{c}_1 - uv\hat{c}_1^\dagger\hat{c}_2^\dagger \\
&- vu\hat{c}_2\hat{c}_1 + v^2\hat{c}_2\hat{c}_2^\dagger\Big) + \Delta\Big(uv\hat{c}_1^\dagger\hat{c}_1 + u^2\hat{c}_1^\dagger\hat{c}_2^\dagger - v^2\hat{c}_2\hat{c}_1 + uv\hat{c}_2\hat{c}_2^\dagger
\end{aligned}$$

$$
\begin{aligned}
&+\ uv\hat{c}_1^\dagger\hat{c}_1 - v^2\hat{c}_1^\dagger\hat{c}_2^\dagger - u^2\hat{c}_2\hat{c}_1 - uv\hat{c}_2\hat{c}_2^\dagger\Big)\\
&= \varepsilon\Big((u^2-v^2)(\hat{c}_1^\dagger\hat{c}_1 - \hat{c}_2\hat{c}_2^\dagger) - 2uv\hat{c}_1^\dagger\hat{c}_2^\dagger - 2uv\hat{c}_2\hat{c}_1\Big)\\
&+\ \Delta\Big(2uv\hat{c}_1^\dagger\hat{c}_1 - 2uv\hat{c}_2\hat{c}_2^\dagger + (u^2-v^2)(\hat{c}_1^\dagger\hat{c}_2^\dagger + \hat{c}_2\hat{c}_)\Big)\\
&= \Big(\varepsilon(u^2-v^2) + \Delta 2uv\Big)(\hat{c}_1^\dagger\hat{c}_1 - \hat{c}_2\hat{c}_2^\dagger)\\
&+\ \Big(-\varepsilon 2uv + \Delta(u^2-v^2)\Big)(\hat{c}_1^\dagger\hat{c}_2^\dagger + \hat{c}_2\hat{c}_1).
\end{aligned}
$$

$$
\text{Let}\qquad u = \cos\theta
$$

$$
v = \sin\theta
$$

which is legitimate since $u^2 + v^2 = 1$.

$$
\Rightarrow\qquad u^2 - v^2 = \cos 2\theta \quad\text{and}\quad 2uv = \sin 2\theta.
$$

Then

$$
\begin{aligned}
\varepsilon(u^2-v^2) + \Delta 2uv &= \varepsilon\cos 2\theta + \Delta\sin 2\theta\\
&= P\cos(2\theta - \alpha)
\end{aligned}
$$

where

$$
P^2 = \varepsilon^2 + \Delta^2 \quad\text{and}\quad \varepsilon = P\cos\alpha,\quad \Delta = P\sin\alpha.
$$

Similarly

$$
-\varepsilon\ 2uv + \Delta(u^2-v^2) = P\sin(2\theta-\alpha).
$$

Under these transformations:

$$
\hat{H} = P\cos(2\theta-\alpha)(\hat{c}_1^\dagger\hat{c}_1 - \hat{c}_2\hat{c}_2^\dagger) + P\sin(2\theta-\alpha)(\hat{c}_1^\dagger\hat{c}_2^\dagger + \hat{c}_2\hat{c}_1).
$$

Under the particular case when $2\theta = \alpha$ which corresponds to $\varepsilon = u^2 - v^2$ and $\Delta = 2uv$,

$$\begin{aligned}\hat{H} &= \sqrt{\varepsilon^2 + \Delta^2}(\hat{c}_1^\dagger \hat{c}_1 - \hat{c}_2 \hat{c}_2^\dagger) \\ &= \sqrt{\varepsilon^2 + \Delta^2}(\hat{c}_1^\dagger \hat{c}_2 + \hat{c}_2^\dagger \hat{c}_2 - 1) \quad \text{as desired.}\end{aligned}$$

(c) the ground state energy is $-\sqrt{\varepsilon^2 + \Delta^2}$. This can be seen by writing $\hat{H}$ in the basis spanned by the states, $|0,0\rangle$, $|0,1\rangle$, $|1,0\rangle$, $|1,1\rangle$, which are written in number-state representation, (e.g. $|1,0\rangle$ corresponds to one particle with momentum $\mathbf{p}_1$ and no particle with momentum $\mathbf{p}_2$). H is diagonal in this basis and is given by,

$$\sqrt{\varepsilon^2 + \Delta^2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

where the states are labeled in he order $\{|0,0\rangle\,,\ |0,1\rangle\,,\ |1,0\rangle\,,\ |1,1\rangle\}$.

(d) Clearly the ground state is a superposition of $|0,0\rangle$ and $|1,1\rangle$.

5. Defining the density matrix for a single particle as,

$$\hat{\rho}_1(\mathbf{x} - \mathbf{y}) = \langle \hat{\psi}^\dagger(\mathbf{x}) \hat{\psi}(\mathbf{y}) \rangle.$$

Express this matrix in the form of creation and annihilation operators.

Answer 5

$$
\begin{aligned}
\psi(\mathbf{y}) &= \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} e^{+i\mathbf{p}\cdot\mathbf{y}} \ \hat{a}_p \\
\psi^\dagger(\mathbf{x}) &= \frac{1}{\sqrt{V}} \sum_{\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{x}} \ \hat{a}_q^\dagger \\
\psi^\dagger(\mathbf{x})\psi(\mathbf{y}) &= \frac{1}{V} \sum_{\mathbf{p},\mathbf{q}} e^{-i\mathbf{q}\cdot\mathbf{x}} \ e^{+i\mathbf{p}\cdot\mathbf{y}} \ \hat{a}_q^\dagger \ \hat{a}_p \\
\langle\psi^\dagger(\mathbf{x})\psi(\mathbf{y})\rangle &= \frac{1}{V} \sum_{\mathbf{p},\mathbf{q}} e^{-i(\mathbf{q}\cdot\mathbf{x}-\mathbf{p}\cdot\mathbf{y})} \ \langle\hat{a}_q^\dagger \ \hat{a}_p\rangle.
\end{aligned}
$$