Solution Assignment 7: Quantum Field Theory

1. In HW 3, Q4(d), we introduced the symmetric energy-momentum tensor for a source-free electromagnetic field

$$\overline{T}^{\mu\nu} = -F^{\mu\beta}\partial^{\nu}A_{\beta} + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \partial_{\lambda}X^{\lambda\mu\nu} \tag{1}$$

where $X^{\lambda\mu\nu} = F^{\mu\lambda} A^{\nu}$. Show that this symmetrized tensor is gauge invariant.[10 marks]

Answer 1

$$\overline{T}^{\mu\nu} = -F^{\mu\beta}\partial^{\nu}A_{\beta} + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \partial_{\lambda}X^{\lambda\mu\nu}$$

where $X^{\lambda\mu\nu} = F^{\mu\lambda} A^{\nu}$.

$$A_{\mu} \mapsto A'_{\mu} = A_{\mu} - \partial_{\mu}\chi$$

Let's look at the terms in $\overline{T}^{\mu\nu}$ one by one.

$$\begin{aligned}
F^{\mu\beta} &= \partial^{\mu}A^{\beta} - \partial^{\beta}A^{\mu} \\
&\mapsto \partial^{\mu}(A^{\beta} - \partial^{\beta}\chi) - \partial^{\beta}(A^{\mu} - \partial^{\mu}\chi) \\
&= \partial^{\mu}A^{\beta} - \partial^{\mu}\partial^{\beta}\chi - \partial^{\beta}A^{\mu} + \partial^{\beta}\partial^{\mu}\chi \\
&= F^{\mu\beta}
\end{aligned}$$

So the electromagnetic field tensor F remains invariant.

$$\begin{aligned}
\partial^{\nu}A_{\beta} &\mapsto \partial^{\nu}(A_{\beta} - \partial_{\beta}\chi) \\
&= \partial^{\nu}A_{\beta} - \partial^{\nu}\partial_{\beta}\chi.
\end{aligned}$$

The second term $-\frac{1}{4}F_{\mu\beta}F^{\mu\beta}$ is clearly invariant.

$$\begin{aligned}
\partial_{\lambda}X^{\lambda\mu\nu} &= \partial_{\lambda}(F^{\mu\lambda}A^{\nu}) \\
&\mapsto \partial_{\lambda}(F^{\mu\lambda}(A^{\nu} - \partial^{\nu}\chi)) \\
&= \partial_{\lambda}(F^{\mu\lambda}A^{\nu}) - \partial_{\lambda}(F^{\mu\lambda}\partial^{\nu}\chi) \\
&= \partial_{\lambda}(F^{\mu\lambda}A^{\nu}) - (\partial_{\lambda}(F^{\mu\lambda})(\partial^{\nu}\chi) - F^{\mu\lambda}(\partial_{\lambda}\partial^{\nu}\chi) \\
&= \partial_{\lambda}(F^{\mu\lambda}A^{\nu}) - F^{\mu\lambda}(\partial_{\lambda}\partial^{\nu}\chi) \quad \text{(using the source free Maxwell's equation)}
\end{aligned}$$

Hence $\overline{T}^{\mu\nu}$ transforms as:

$$\begin{aligned} &- \; F^{\mu\beta}(\partial^\nu A_\beta - \partial^\nu\partial_\beta\chi) + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \partial_\lambda X^{\lambda\mu\nu} - F^{\mu\lambda}(\partial_\lambda\partial^\nu\chi) \\ &= \overline{T}^{\mu\nu} + F^{\mu\beta}\partial^\nu\partial_\beta\chi - F^{\mu\beta}\partial_\beta\partial^\nu\chi \\ &= \overline{T}^{\mu\nu} \end{aligned}$$

Hence $\overline{T}^{\mu\nu}$ is gauge invariant.

2. (a) For the source-free Proca Lagrangian (corresponding to a massive vector field)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu \tag{2}$$

the energy-momentum tensor is given by,

$$\overline{T}^{\mu\nu} = -F^{\mu\beta}\partial^\nu A_\beta + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \partial_\lambda(F^{\mu\lambda A^\nu}) - \frac{1}{2}m^2 g^{\mu\nu}A_\alpha A^\alpha. \tag{3}$$

Compute the energy density in terms of **E**, **B**, A_0 and **A**. [10 marks]

(b) Now determine the momentum density in terms of these vectors. [10 marks]

Answer 2

(a)

$$\begin{aligned} \overline{T}^{\mu\nu} &= -F^{\mu\beta}\partial^\nu A_\beta + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \partial_\lambda(F^{\mu\lambda A^\nu}) - \frac{1}{2}m^2 g^{\mu\nu}A_\alpha A^\alpha \\ \overline{T}^{oo} &= -F^{o\beta}\partial^o A_\beta - \frac{1}{2}\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right) + \partial_\lambda(F^{o\lambda}A^o) - \frac{1}{2}m^2 g^{\mu\nu}A_\alpha A^\alpha \end{aligned}$$

Now

$$\begin{aligned} A_\alpha A^\alpha &= (A^o)^2 - |\mathbf{A}|^2 \\ F^{o1} &= -E^i, \quad F^{oo} = 0 \end{aligned}$$

$\partial_\lambda(F^{o\lambda}A^o)$ becomes,

$$\begin{aligned} -\partial_i(E^i A^o) &= (-\partial_i E^i)A^o - E^i\partial_i A^o \\ &= -\nabla\cdot\mathbf{E}\;A^o - \mathbf{E}\cdot\nabla\;A^o \\ -F^{o\beta}\partial^o A_\beta &= -F^{oi}\partial^o A_i = E^i\partial^o A_i = -\mathbf{E}\cdot\dot{\mathbf{A}}. \end{aligned}$$

Inserting all these values into $\overline{T}^{oo}$ tields:

$$\overline{T}^{oo} = -\mathbf{E}\cdot\dot{\mathbf{A}} - \frac{1}{2}\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right) - \nabla\cdot\mathbf{E}\ A^o - \mathbf{E}\cdot\nabla\ A^o - \frac{1}{2}m^2\left((A^o)^2 - |\mathbf{A}|^2\right)$$

Keep in mind $\nabla\cdot\mathbf{E} = 0$ (no sources), simplify:

$$\overline{T}^{oo} = -\mathbf{E}\cdot(\dot{\mathbf{A}} + \nabla A^o) - \frac{1}{2}\left(|\mathbf{E}|^2 - |\mathbf{B}|^2\right) - \frac{1}{2}m^2(A^o)^2 + \frac{1}{2}m^2|\mathbf{A}|^2$$

$$\text{Now}\quad \mathbf{E} = -\dot{\mathbf{A}} - \nabla A^o$$

So,

$$\overline{T}^{oo} = \frac{1}{2}|\mathbf{E}|^2 + \frac{1}{2}|\mathbf{B}|^2 + \frac{1}{2}m^2|\mathbf{A}|^2 - \frac{1}{2}m^2(A^o)^2.$$

(b)

$$\overline{T}^{\mu\nu} = -F^{\mu\beta}\partial^\nu A_\beta + \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} + \partial_\lambda(F^{\mu\lambda A^\nu}) - \frac{1}{2}m^2 g^{\mu\nu}A_\alpha A^\alpha$$

Put $\mu = 0$ and $\nu = i$ in above equation,

$$\overline{T}^{oi} = -F^{o\beta}\partial^i A_\beta + \frac{1}{4}g^{oi}F_{\alpha\beta}F^{\alpha\beta} + \partial_\lambda(F^{o\lambda}A^i) - \frac{1}{2}m^2 g^{oiA_\alpha A^\alpha}$$

Now $F^{oi} = -E^i$, $F^{oo} = 0$ and $g^{oi} = 0$. Therefore

$$\begin{aligned}\overline{T}^{oi} &= E^j\partial^i A_j + \partial_j(F^{oj}A^i)\\ &= E^j\partial^i A_j + \partial_j(E^j A^i)\\ &= E^j\partial^i A_j - (\partial_j E^j)A^i - E^j(\partial_j A^i)\\ &= E^j(\partial^i A_j - \partial_j A^i) \quad (\text{since } \boldsymbol{\Delta}\cdot\mathbf{E} = 0).\end{aligned}$$

Now consider

$$\begin{aligned}(\mathbf{E}\times\mathbf{B})^i &= (\mathbf{E}\times\boldsymbol{\Delta}\times\mathbf{A})^i.\\ \overline{T}^{oi} &= \epsilon^{ijk}E_j B_k\\ &= \epsilon^{ijk}E_j(\boldsymbol{\Delta}\times\mathbf{A})_k\\ &= \epsilon^{ijk}E_j(\epsilon_{k\ell m}\partial^\ell A^m)\\ &= \epsilon^{kij}\epsilon_{k\ell m}(E_j\partial^\ell A^m) \quad (\text{I can cyclically permute } i,\ j,\ k).\\ &= (\delta^i_\ell\delta^j_m - \delta^i_m\delta^j_\ell)(E_j\partial^\ell A^m)\\ &= E_j\partial^i A^j - E_j\partial^j A_i\\ &= E^j\partial^i A_j - E^j\partial_j A^i\end{aligned}$$

$$\text{Hence}\quad \overline{T}^{oi} = (\mathbf{E}\times\mathbf{B})^i \text{ is the momentum density.}$$

3. Consider an undamped harmonic oscillator. It's Green's function is defined as

$$\left(m\frac{\partial^2}{\partial t^2} + mE_0^2\right)G(t,u) = \delta(t-u). \tag{4}$$

Use the following definitions of Fourier transformation and its inverse:

$$\begin{aligned}\tilde{F}(E) &= \int_{-\infty}^{\infty} dt\ f(t)\ e^{iEt}\\ f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dE\tilde{F}(E)e^{iEt}.\end{aligned}$$

(a) Fourier transform Equation (4) above to obtain $\tilde{G}(E,u)$. [3 marks]

(b) Inverse Fourier transform $\tilde{G}(E,u)$ and hence determine the retarded Green function $G(t,u)$. Notes: (i) Sketch the function. (ii) You will need to perform contour integration in the complex plane, employing Cauchy's theorem and the reside theorem. Furthermore, you will ensure causality by adding infinitesimal imaginary components to the energy E. Clearly show all your working. [17 marks]

Answer 3

(a) Fourier transforming both sides and using the property

$$f\left(\frac{\partial^2}{\partial t^2}\right) = (iE)^2 = -E^2$$

We have,

$$\begin{aligned}(-mE^2 + mE_0^2)G(E,u) &= \int_{-\infty}^{\infty} dt\ e^{iEt}\ \delta(t-u)\\ &= e^{iEu}.\end{aligned}$$

So,

$$G(E,u) = \frac{e^{iEu}}{-m(E^2 - E_0^2)}$$

(b)

$$\begin{aligned}G(t,u) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dE\ G(E,u)\ e^{-iEt}\\ &= -\frac{1}{2\pi m}\int_{-\infty}^{\infty}\frac{dE\ e^{-iE(t-u)}}{(E^2 - E_0^2)}.\end{aligned}$$

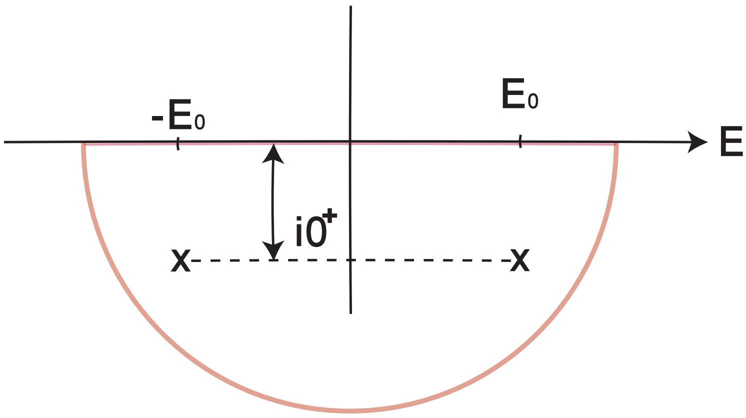
In order to ensure casuality, I like to replace $E \mapsto E + i0^+$, where 0^+ is an infinitesimal positive number.

$$G(t,u) = -\frac{1}{2\pi m}\int_{-\infty}^{\infty} \frac{dE\ e^{-i(E-i0^+)(t-u)}}{((E+i0^+)^2 - E_0^2)}.$$

We now use results from complex integration to solve this. Let,

$$f(E) = -\frac{1}{2\pi m}\ \frac{e^{-i(E-i0^+)(t-u)}}{((E+i0^+)+E_0)((E+i0^+)-E_0)}$$

be the integrand here. The poles of $F(E)$ are at $E_1 = -E_0 - i0^+$ and $E_2 = E_0 - i0^+$. These poles are shown in the complex plane as lying in the lower half plane.



We draw a closed contour with an infinitely large semi-circle in the lower half plane. Using the residue theorem applied to the contour shown, we have:

$$G(t,u) + \int f(z)\ dz = -2\pi i\ (\text{sum of residues of } f(z)) \tag{5}$$

The minus sign on the R.H.S shows that we go around the circle in a clockwise fashion. On the semicircle $E = -i|\eta|$, so

$$\begin{aligned} f(E) &= -\frac{1}{2\pi m}\ \frac{e^{-i(-i|\eta|-i0^+)(t-u)}}{((E-i0^+ + E_0)((E-i0^+ - E_0)} \\ &= -\frac{1}{2\pi m}\ \frac{e^{(-|\eta|-i0^+)(t-u)}}{((E-i0^+ + E_0)((E-i0^+ - E_0)} \end{aligned}$$

The desired $G(t,u)$ is the integral of $f(z)$ along the real line from $-\infty$ to $+\infty$.

$$\begin{aligned} \oint f(z)\ dz &= G(t,u) + \int_{\text{semicircle}} -\frac{1}{2\pi m}\ \frac{e^{(-|\eta|-i0^+)(t-u)}}{((E-i0^+ + E_0)((E-i0^+ - E_0)} \\ &= -2\pi i(\text{sum of residues}). \end{aligned}$$

For $t > u$, the integral over the semiinfinite semicircle is zero. So,

$$G(t,u) = -2\pi i(\text{sum of residues})$$

Residue of $f(E)$ at $E_1 = -E_0 - i0^+$

$$\begin{aligned}
&= -\frac{1}{2\pi m} \lim_{E\to -E_0 - i0^+} (E + E_0 + i0^+) \frac{e^{-i(E-i0^+)(t-u)}}{((E+i0^+ + E_0)((E+i0^+ - E_0)} \\
&= -\frac{1}{2\pi m} \frac{e^{-i(-E_0-i0^+-i0^+)(t-u)}}{((-E_0 - i0^+ + i0^+ - E_0)} \\
&= -\frac{1}{2\pi m} \frac{e^{i(E_0+2i0^+)(t-u)}}{(-2E_0)}.
\end{aligned}$$

Residue of $f(E)$ at $E_2 = E_0 - i0^+$

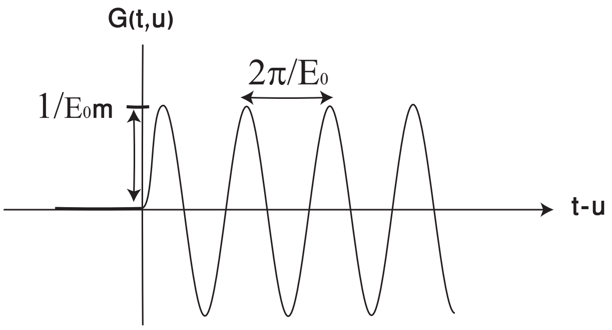
$$\begin{aligned}
&= -\frac{1}{2\pi m} \lim_{E\to E_0 - i0^+} (E - E_0 + i0^+) \frac{e^{-i(E-i0^+)(t-u)}}{((E+i0^+ + E_0)((E+i0^+ - E_0)} \\
&= -\frac{1}{2\pi m} \frac{e^{-i(E_0-i0^+-i0^+)(t-u)}}{((E_0 - i0^+ + i0^+ + E_0)} \\
&= -\frac{1}{2\pi m} \frac{e^{-i(E_0-2i0^+)(t-u)}}{(2E_0)}.
\end{aligned}$$

$$\begin{aligned}
\therefore \quad G(t,u) \text{ for } t > u &= \left(-\frac{2\pi i}{2E_0}\right)\left(-\frac{1}{2\pi m}\right)\big(-e^{iE_0(t-u)} + e^{iE_0(t-u)}\big) \\
&= \frac{i}{2E_0 m}\big(-e^{iE_0(t-u)} + e^{iE_0(t-u)}\big) \\
&= \frac{1}{E_0 m}\,\frac{1}{2i}\big(-e^{iE_0(t-u)} + e^{iE_0(t-u)}\big) \\
&= \frac{1}{E_0 m}\sin\big(E_0(t-u)\big).
\end{aligned}$$

Likewise for $t < u$, we can draw an infinite semi-circle in the upper half plane. Since it does not enclose any poles, each side of equation (5) is $\oint f(z)\ dz = 0$. On the semi-circle,$|E| = +i|\eta|$ yielding the integral on the semi-circle to be zero as well. This implies $G(t,u) = 0$. Hence

$$\begin{aligned}
G(t,u) &= \frac{1}{E_0 m}\sin\big(E_0(t-u)\big) \quad \text{for} \quad t > u, \\
&= 0 \quad \text{for} \quad t < u.
\end{aligned}$$

Here is a sketch.


G(t,u)
2π/E0
1/E0m
t-u