Solution Assignment 8: Quantum Field Theory

1. Show a neat derivation of the Heaviside function expanded as an integral,

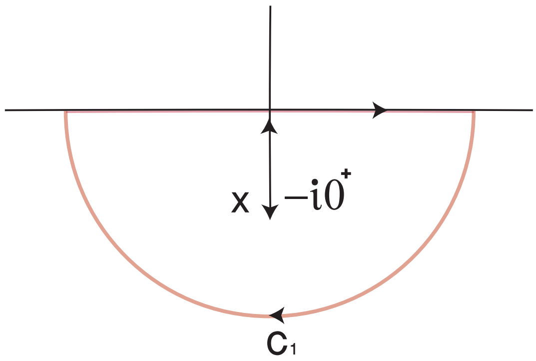
$$\Theta(t) = \frac{i}{2\pi}\int_{-\infty}^{\infty} dz\ \frac{e^{-izt}}{z + i0^{+}}.$$

[10 **marks**]

Answer 1

We need to show that $\Theta(t) = 1$ for $t > 0$ and $\Theta(t) = 0$ for $t < 0$.

Case I**:** Let $t > 0$



On the semi-circle drawn in the (lower half plane) L.H.P, z is large, negative and imaginary, i.e., $z = -i|\eta|$. Substituting this in the candidate integral,

$$i\int_{\text{semicircle } C_1} \frac{dz}{2\pi}\frac{e^{-i(-i|\eta|)t}}{z + i0^{+}} = i\int_{\text{semicircle } C_1} \frac{dz}{2\pi}\frac{e^{-|\eta|t}}{z + i0^{+}} = 0$$

Now let's integrate over the real axis as well as the semicircle C_1. The integral over C_1 is zero, therefore

$$i\int_{-\infty}^{\infty}\frac{dz}{2\pi}\frac{e^{-izt}}{z + i0^{+}} + 0 \;=\; -2\pi i\ \mathrm{Res}\big(f(-i0^{+})\big)$$

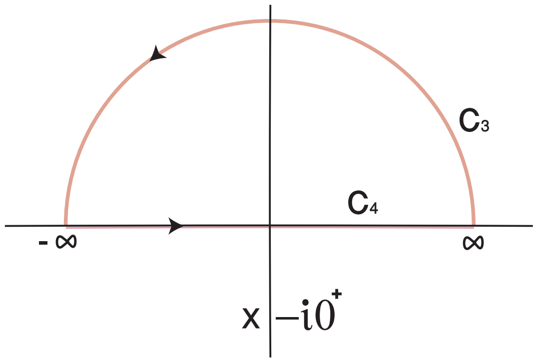
where $-i0^{+}$ is the position of the pole. Now

$$\begin{aligned}\mathrm{Res}\big(f(-i0^{+}\big) \;&=\; \frac{i}{2\pi}\lim_{z\to -i0^{+}}(z + i0^{+})\frac{e^{-izt}}{(z + i0^{+})}\\ &=\; \frac{ie^{-0^{+}t}}{2\pi}\end{aligned}$$

$$\therefore\quad \frac{i}{2\pi}\int_{-\infty}^{\infty} dz\ \frac{e^{-izt}}{z + i0^{+}} \;=\; -2\pi i\frac{ie^{-0^{+}t}}{2\pi} = e^{-0^{+}t} = 1 \quad (\text{as } 0^{+}\to 0).$$

Hence for $t > 0$, the integral is zero.

Case II**:** Let $t < 0$



For this case, we draw a semicircle in the upper half plane (U.H.P) and it does not enclose the pole. Along the semicircle C_3, we have a large positive η,

$$
\begin{aligned}
z &= +i|\eta| \\
\therefore \quad \frac{i}{2\pi}\int_{C_3} dz\, \frac{e^{-i(i|\eta|t)}}{z+i0^+} &= \frac{i}{2\pi}\int_{C_3} dz\, \frac{1}{z+i0^+}\, e^{i|\eta|t} \qquad (\text{for} \quad t<0)
\end{aligned}
$$

Now integrating over the contour C_3 and C_4, we obtain

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{i}{2\pi}\, \frac{e^{izt}}{z+i0^+} + 0 &= 0 \\
\text{Hence} \quad i\int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{e^{izt}}{z+i0^+} &= 0 \qquad (\text{for} \quad t<0)
\end{aligned}
$$

Since the contour $C_3 + C_4$ does not enclose the plane. Combining the result from the two cases, proves the desired identity.

2. Is

$$\Delta(x,y) = \int \frac{d^4p}{(2\pi)^4}\, \frac{i}{p^2 - m^2 + i0^+} e^{-ip\cdot(x-y)}$$

the Green function for the Klein-Gordon operator? Show your working. [5 **marks**]

Answer 2

$$
\begin{aligned}
(\partial^2 + m^2)\Delta(x,y) &= (\partial_t^2 - \nabla^2 + m^2)\Delta(x,y) \\
\therefore \quad (\partial^2 + m^2)\Delta(x,y) &= \int \frac{d^4p}{(2\pi)^4}\, \frac{i}{p^2 - m^2 + i0^+} e^{-ip\cdot(x-y)}(-(p^0)^2 + |\mathbf{p}|^2 + m^2) \\
&= -i\int \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-y)} \qquad (\text{since} \quad p^2 = (p^0)^2 + |\mathbf{p}|^2 = m^2) \\
&= -i\delta^4(x-y) \\
\text{Hence} \quad \Delta(x,y) &= \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\, \frac{i}{p^2 - m^2 + i0^+}
\end{aligned}
$$

is a Green's function for the Klein-Gordon operator.

3. In class, we derived the following expression for the free propagation for scalar fields:

$$\Delta(x,y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\, \frac{i}{(p^0)^2 - E_{\mathbf{p}}^2}$$

Replace $E_{\mathbf{p}}$ with $E_{\mathbf{p}} - i0^+$, where 0^+ is an infinitesimal positive number. Compute the integral given above using the rules of contour integration, verifying in the process, that

$$\Delta(x,y) = \int \frac{d^3p}{(2\pi)^3}\Big(\Theta(x^0-y^0)e^{-ip\cdot(x-y)} + \Theta(y^0-x^0)e^{+ip\cdot(x-y)}\Big).$$

What is the role of the minute term $i0^+$? [15 **marks**]

Answer 3

$$\Delta(x,y) = \int \frac{d^4p}{(2\pi)^4}\, e^{-ip\cdot(x-y)}\, \frac{i}{(p^0)^2 - E_{\mathbf{p}}^2}.$$

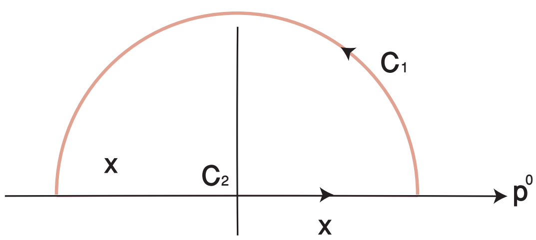
Let's look at $\dfrac{1}{(p^0)^2 - E_{\mathbf{p}}^2}$

$$\frac{1}{(p^0)^2 - E_{\mathbf{p}}^2} = \frac{1}{(p^0 + E_{\mathbf{p}})(p^0 - E_{\mathbf{p}})}$$

Replace $E_{\mathbf{p}}$ with $E_{\mathbf{p}} - i0^+$. This produces two poles, one in the L.H.P (at $E_{\mathbf{p}} - i0^+$) and other in the U.H.P (at $-E_{\mathbf{p}} + i0^+$). Say,

$$f(p^0) = \frac{e^{-ip^0(x^0-y^0)}}{(p^0 + E_{\mathbf{p}} - i0^+)(p^0 - E_{\mathbf{p}} + i0^+)}.$$

We draw a semi-infinite circle C_1 in the upper half plane that encloses the pole at $-E_{\mathbf{p}} + i0^+$.



Over this contour, we have from the residue theorem,

$$\int_{-\infty}^{\infty} dp^0\; f(p^0) + \int_{\text{semicircle}} dp^0\; f(p^0) = 2\pi i\,\big(\text{Residue of } f(p^0) \text{ at } -E_{\mathbf{p}} + i0^+\big). \quad (1)$$

The first integral over the real p^0 axis is the desired, while over the semicircle,

$$p^0 = i|\eta|,$$

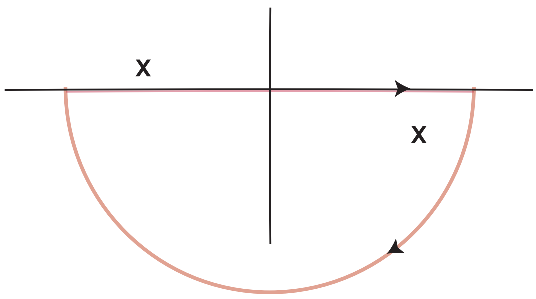
yielding

$$f(p^0) = \frac{e^{+i|\eta|(x^0-y^0)}}{(i|\eta| + E_{\mathbf{p}} - i0^+)(i|\eta| - E_{\mathbf{p}} + i0^+)}\,.$$

If $x^0 < y^0$, the integral vanishes when $|\eta| \to \infty$. So, equation (1) becomes,

$$\begin{aligned}
\int_{-\infty}^{\infty} dp^0\; f(p^0) &= 2\pi i \lim_{p^0 \to -E_{\mathbf{p}}+i0^+} \frac{(p^0 + E_{\mathbf{p}} - i0^+)e^{-ip^0(x^0-y^0)}}{(p^0 + E_{\mathbf{p}} - i0^+)(p^0 - E_{\mathbf{p}} + i0^+)} \\
&= 2\pi i \frac{e^{-i(-E_{\mathbf{p}}+i0^+)(x^0-y^0)}}{-E_{\mathbf{p}} + i0^+ - E_{\mathbf{p}} + i0^+} \\
&= -2\pi i \frac{e^{iE_{\mathbf{p}}(x^0-y^0)}}{2E_{\mathbf{p}}} \\
&= -\pi i e^{iE_{\mathbf{p}}(x^0-y^0)}.
\end{aligned}$$

We now draw a semi-infinite circle over the L.H.P. encircling the other pole at $E_{\mathbf{p}} - i0^+$.



Employing equation (1)

$$\begin{aligned}
\int_{-\infty}^{\infty} dp^0\; f(p^0) + \int dp^0\; f(p^0) &= -2\pi i \,\big(\text{Residue of } f(p^0) \text{ at } E_{\mathbf{p}} - i0^+\big). \\
\text{Now} \quad p^0 &= -i|\eta|, \quad \text{so,} \\
f(p^0) &= \frac{e^{-|\eta|}(x^0-y^0)}{(-i|\eta| + E_{\mathbf{p}} - i0^+)(-i|\eta| - E_{\mathbf{p}} + i0^+)}
\end{aligned}$$

whose integral vanishes when $x^0 > y^0$. Hence, equation (1) works out as

$$\begin{aligned}
\int_{-\infty}^{\infty} dp^0\; f(p^0) &= -2\pi i \lim_{p^0 \to E_{\mathbf{p}}-i0^+} \frac{(p^0 - E_{\mathbf{p}} + i0^+)e^{-ip^0}(x^0-y^0)}{(p^0 + E_{\mathbf{p}} - i0^+)(p^0 - E_{\mathbf{p}} + i0^+)} \\
&= -2\pi i \frac{e^{-iE_{\mathbf{p}}}(x^0-y^0)}{2E_{\mathbf{p}}} \\
&= -\pi i \frac{e^{-iE_{\mathbf{p}}}(x^0-y^0)}{E_{\mathbf{p}}}.
\end{aligned}$$

Putting these results together,

$$\int dp^0 \; \frac{e^{-ip^0(x^0-y^0)}}{(p^0+E_{\mathbf{p}}-i0^+)(p^0-E_{\mathbf{p}}+i0^+)} \;=\; \frac{-\Theta(x^0-y^0)\pi i e^{-iE_{\mathbf{p}}}(x^0-y^0)}{E_{\mathbf{p}}}$$
$$= \frac{-\Theta(y^0-x^0)\pi i e^{+iE_{\mathbf{p}}}(x^0-y^0)}{E_{\mathbf{p}}}$$

$$\begin{aligned}\Delta(x,y) &= \int \frac{d^3p}{(2\pi)^4} i e^{+\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\left(-\frac{\pi i}{E_{\mathbf{p}}}\right)\left(\Theta(x^0-y^0)e^{-iE_{\mathbf{p}}(x^0-y^0)}+\Theta(y^0-x^0)e^{+iE_{\mathbf{p}}(x^0-y^0)}\right) \\ &= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}\left(\Theta(x^0-y^0)e^{-ip\cdot(x-y)}+\Theta(y^0-x^0)e^{+ip\cdot(x-y)}\right.\end{aligned}$$

4. (a) For a massive scalar field, the Lagrangian density is:

$$\mathscr{L} = \frac{1}{2}\Big(\partial_\mu\phi(x)\Big)^2 - \frac{1}{2}m^2\Big(\phi(x)\Big)^2.$$

Express the action $S = \int d^4x \; L$ in the momentum space and comment how S depends on the momentum representation of the Feynmann propagator for the scalar field. [10 **marks**]

(b) Now let's discretize the scalar field by positioning it on an equally spaced chain in one dimension. The discretization is achieved by:

$$\phi_j(t) = \frac{1}{\sqrt{L}}\sum_p \int \frac{d\omega}{2\pi}\; \widetilde{\phi}_p \; e^{-i(\omega t - pja)}$$

where $L = Na$ is the length of the chain, j is the spatial index and $\widetilde{\phi}_p(\omega)$ is the Fourier transform of $\phi_j(t)$. Using

$$\frac{1}{L}\sum_j e^{+i(p+q)ja} = \delta^{(1)}(p+q)$$

and the fact that we have dealing with the $(1+1)$ Minkonski space, derive the action S in momentum space.

Using results from the previous question, state the Green Function for this one-dimensional chain. The excitations of such a field are called phonons.

[15 **marks**]

Answer 4

(a)

$$\begin{aligned}
\mathscr{L} &= \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2(\phi)^2 \\
S &= \int d^4x\ \mathscr{L} = S_1 + S_2 \\
S_1 &= \frac{1}{2}\int d^4x\ (\partial_\mu\phi)(\partial^\mu\phi) \\
\text{and} \quad S_2 &= -\frac{1}{2}\int d^4x\ m^2(\phi)^2
\end{aligned}$$

Now

$$\begin{aligned}
\phi(x) &= \int \frac{d^4p}{(2\pi)^4}\ \widetilde{\phi}(p)\ e^{-ip\cdot x} \qquad \text{(expressing in the Fourier domain)} \\
\partial_\mu\phi(x) &= \int \frac{d^4p}{(2\pi)^4}\ \widetilde{\phi}(p)\ (-ip_\mu)\ e^{-ip\cdot x} \\
S_1 &= -\frac{1}{2}\int \frac{d^4x\ d^4p\ d^4q}{(2\pi)^4(2\pi)^4}\ \widetilde{\phi}(p)\ \widetilde{\phi}(q)\ p\cdot q\ e^{-i(p+q)\cdot x}
\end{aligned}$$

Since

$$\frac{1}{(2\pi)^4}\int d^4x\ e^{-i(p+q)\cdot x} = \delta^{(4)}(p+q)$$

$$\begin{aligned}
S_1 &= -\frac{1}{2}\int \frac{d^4p\ d^4q}{(2\pi)^4}\ \widetilde{\phi}(p)\ \widetilde{\phi}(q)\ p\cdot q\ \delta^{(4)}(p+q) \\
&= \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\ \widetilde{\phi}(p)\ \widetilde{\phi}(-p)\ p^2
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
S_2 &= -\frac{1}{2}\int d^4x\ m^2\big(\phi(x)\big)^2 \\
&= -\frac{m^2}{2}\int \frac{d^4x\ d^4p\ d^4q}{(2\pi)^4(2\pi)^4}\ \widetilde{\phi}(p)\ \widetilde{\phi}(q)\ e^{-i(p+q)\cdot x} \\
&= -\frac{m^2}{2}\int \frac{d^4p\ d^4q}{(2\pi)^4}\ \widetilde{\phi}(p)\ \widetilde{\phi}(q)\ \delta^{(4)}(p+q) \\
&= -\frac{m^2}{2}\int \frac{d^4p}{(2\pi)^4}\ \widetilde{\phi}(p)\ \widetilde{\phi}(-p)
\end{aligned}$$

Hence

$$S = S_1 + S_2 = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\ \widetilde{\phi}(p)\ \widetilde{\phi}(-p)(p^2 - m^2)$$

Since $\widetilde{G}(p) = \dfrac{i}{p^2 - m^2}$.

$$S = \frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\ \widetilde{\phi}(p)\ \widetilde{\phi}(-p)\left(\frac{i}{\widetilde{G}(p)}\right) \tag{2}$$

which is the desired momentum space representation of the action.

(b) Upon discretizing,

$$\begin{aligned} S &= \int dt\ dx\ \mathscr{L} \\ &= \frac{1}{2}\int dt\ dx\ \Big[(\partial_0\phi)^2 - (\partial_1\phi)^2\Big] = \frac{1}{2}\int dt \sum_j \end{aligned}$$

Now

$$\phi_j(t) = \frac{1}{\sqrt{L}}\sum_p \int \frac{d\omega}{2\pi}\widetilde{\phi}_p(\omega)\ e^{-i(\omega t - pja)}$$

In this expression ω and p are conjugate to time and the position respectively.

$$\begin{aligned} (\partial_0\phi) &= -i\frac{1}{L}\sum_p \int \frac{d\omega}{2\pi}\ \widetilde{\phi}_p(\omega)\ \omega\ e^{-i(\omega t - pja)} \\ (\partial_0\phi)^2 &= +\frac{1}{L}\sum_{p,q}\int \frac{d\omega\ d\nu}{(2\pi)^2}\ \widetilde{\phi}_p(\omega)\ \widetilde{\phi}_q(+\nu)\ \omega\nu\ e^{-i((\omega+\nu)t - pja)}\ e^{+i(p+q)ja} \\ S_1 &= \frac{1}{2}\int dt\ dx\ (\partial_0\phi)^2 \\ &= +\frac{1}{2L}\int dt\ \sum_{i,p,q}\int \frac{d\omega\ d\nu}{(2\pi)^2}\ \widetilde{\phi}_p(\omega)\ \widetilde{\phi}_q(-\nu)\ \omega\nu\ e^{-i((\omega+\nu)t - pja)}\ e^{+i(p+q)ja} \\ &= +\frac{1}{2}\int dt \sum_p \int \frac{d\omega\ d\nu}{(2\pi)^2}\ \widetilde{\phi}_p(\omega)\ \widetilde{\phi}_{-p}(\omega)\ \omega\nu e^{-i(\omega p\nu)} \\ S_1 &= +\frac{1}{2}\sum_p \int \frac{d\omega}{2\pi}\ \widetilde{\phi}_p(\omega)\ \widetilde{\phi}_{-p}(\omega)\ \omega^2 \end{aligned}$$

where we have used

$$\begin{aligned} \frac{1}{L}\sum_j e^{+i(p+q)ja} &= \delta^{(2)}(p+q), \\ \text{and} \qquad \int \frac{dt}{2\pi}e^{-i(\omega+\nu)t} &= \delta^{(1)}(\omega+\nu) \end{aligned}$$

Likewise

$$\begin{aligned} S_2 &= -\frac{m^2}{2}\int dt \sum_j (\partial_1\phi)^2 \\ &= -\frac{m^2}{2}\int dt \sum_j \left(\frac{\phi_{j+1}-\phi_j}{a}\right)^2 \\ &= -\frac{m^2}{2a^2}\int dt \sum_j \frac{1}{L}\sum_{pq}\int \frac{d\omega}{2\pi}\ \frac{d\nu}{2\pi}\left(\widetilde{\phi}_p(\omega)\ e^{-i(\omega t - p(j+1)a)} - \widetilde{\phi}_p(\omega)\ e^{-i(\omega t - pja)}\right) \\ &\left(\widetilde{\phi}_q(\nu)\ e^{-i(\nu t - q(j+1)a)} - \widetilde{\phi}_q(\nu)\ e^{-i(\nu t - pja)}\right) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{m^2}{2a^2L}\int dt \sum_{j,p,q}\int \frac{d\omega\ d\nu}{(2\pi)^2}\left[e^{+ipja}e^{-i\omega t}\widetilde{\phi}_p(\omega)(e^{+ipa}-1) - e^{+ipja}e^{-i\nu t}\widetilde{\phi}_q(\omega)(e^{+iqa}-1)\right] \\
&= -\frac{m^2}{2a^2L}\int dt \sum_{j,p,q}\int \frac{d\omega\ d\nu}{(2\pi)^2}\left[e^{+i(p+q)ja}e^{-i(\omega+\nu)t}\widetilde{\phi}_p(\omega)\widetilde{\phi}_q(\nu)(e^{+i(p+q)a} - e^{ipa} - e^{iqa} + 1)\right] \\
&= -\frac{m^2}{2a^2}\int dt \sum_{p,q}\int \frac{d\omega\ d\nu}{(2\pi)^2}\ \delta_{p,-q}e^{+i(p+q)ja}e^{-i(\omega+\nu)t}\widetilde{\phi}_p(\omega)\widetilde{\phi}_q(\omega)(e^{+i(p+q)a} - e^{ipa} - e^{iqa} + 1) \\
&= -\frac{m^2}{2a^2}\int dt \sum_{p}\int \frac{d\omega\ d\nu}{(2\pi)^2}\ e^{-i(\omega+\nu)t}\widetilde{\phi}_p(\omega)\widetilde{\phi}_{-p}(\nu)(2-2\cos pa) \\
&= -\frac{m^2}{a^2}\sum_{p}\int \frac{d\omega\ d\nu}{(2\pi)^2}\ \delta^{(1)}(\omega+\nu)e^{-i(\omega+\nu)t}\widetilde{\phi}_p(\omega)\widetilde{\phi}_{-p}(\nu)(1-\cos pa) \\
&= -\frac{m^2}{a^2}\sum_{p}\int \frac{d\omega}{2\pi}\ \widetilde{\phi}_p(\omega)\ \widetilde{\phi}_{-p}(\omega)(1-\cos pa) \\
S &= S_1 + S_2 \\
&= \frac{1}{2}\sum_p\int\frac{d\omega}{2\pi}\ \widetilde{\phi}_p(\omega)\ \widetilde{\phi}_{-p}(\omega)\left[\omega^2 - \frac{2m^2}{a^2}(1-\cos pa)\right] \\
&= \frac{1}{2}\sum_p\int\frac{d\omega}{2\pi}\ \widetilde{\phi}_p(\omega)\ \widetilde{\phi}_{-p}(\omega)\left[\frac{\omega^2a^2 - 2m^2(1-\cos pa)}{a^2}\right] \\
&= \frac{1}{2}\sum_p\int\frac{d\omega}{2\pi}\ \widetilde{\phi}_p(\omega)\ \widetilde{\phi}_{-p}(\omega)\left(\omega^2 - \frac{2m^2}{a^2}(1-\cos pa)\right)
\end{aligned}
$$

Hence by comparing with Eq. (2) we obtain,

$$
\widetilde{G}(p) = \frac{i}{\omega^2 - \frac{2m^2}{a^2}(1-\cos pa)}\ .
$$

5. Show that *only* if $\hat{H}_{1I}(t)$ is self-commuting at all times, does

$$
\hat{U}(t_2,t_1) = e^{-i\int_{t_1}^{t_2}\hat{H}_{1I}(\tau)\ d\tau}
$$

represent a solution of

$$
i\frac{d}{dt_2}\hat{U}(t_2,t_1) = \hat{H}_{1I}\ \hat{U}(t_2,t_1).
$$

[10 **marks**]

Answer 5

We are given that

$$
\hat{U}(t_2,t_1)\ =\ e^{-i\int_{t_1}^{t_2}\hat{H}_{1I}(\tau)d\tau}
$$

Using definition of derivative,

$$\begin{aligned}
\frac{d}{dt_2}\hat{U}(t_2,t_1) &= \lim_{\Delta\tau\to 0}\frac{\hat{U}(t_2+\Delta\tau,t_1)-\hat{U}(t_2,t_1)}{\Delta\tau} \\
&= \lim_{\Delta\tau\to 0}\left(\frac{e^{-i\int_{t_1}^{t_2+\Delta\tau}\hat{H}_{1I}(\tau)d\tau}-e^{-i\int_{t_1}^{t_2}\hat{H}_{1I}(\tau)d\tau}}{\Delta\tau}\right)
\end{aligned}$$

$$= \lim_{\Delta\tau\to 0}\frac{1}{\Delta\tau}\left(\frac{e^{-i\int_{t_1}^{t_2+\Delta\tau}\hat{H}_{1I}(\tau)d\tau}\ e^{-i\int_{t_1}^{t_2}\hat{H}_{1I}(\tau)d\tau}-e^{-i\int_{t_1}^{t_2}\hat{H}_{1I}(\tau)d\tau}}{\Delta\tau}\right) \tag{3}$$

This preceding step is true only if $\hat{H}_{1I}(\tau)$ is self commuting. Continuing with the derivative,

$$\begin{aligned}
&= \lim_{\Delta\tau\to 0}\frac{1}{\Delta\tau}\left(e^{-i\int_{t_1}^{t_2+\Delta\tau}\hat{H}_{1I}(\tau)d\tau}-1\right)e^{-i\int_{t_1}^{t_2}\hat{H}_{1I}(\tau)d\tau} \\
&= \lim_{\Delta\tau\to 0}\frac{1}{\Delta\tau}\left(1-i\Delta\tau\hat{H}_{1I}(\tau)d\tau+\frac{(-i\Delta\tau\hat{H}_{1I}(\tau)d\tau)^2}{2!}+\ldots-1\right)e^{-i\int_{t_1}^{t_2}\hat{H}_{1I}(\tau)d\tau} \\
&= -i\hat{H}_{1I}(\tau)\hat{U}(t_2,t_1) \\
\therefore \quad i\frac{d}{dt_2}\hat{U}(t_2,t_1) &= +\hat{H}_{1I}(\tau)\hat{U}(t_2,t_1)
\end{aligned}$$

is true provided we can breakup the propagation as we have done in Eq. (3).