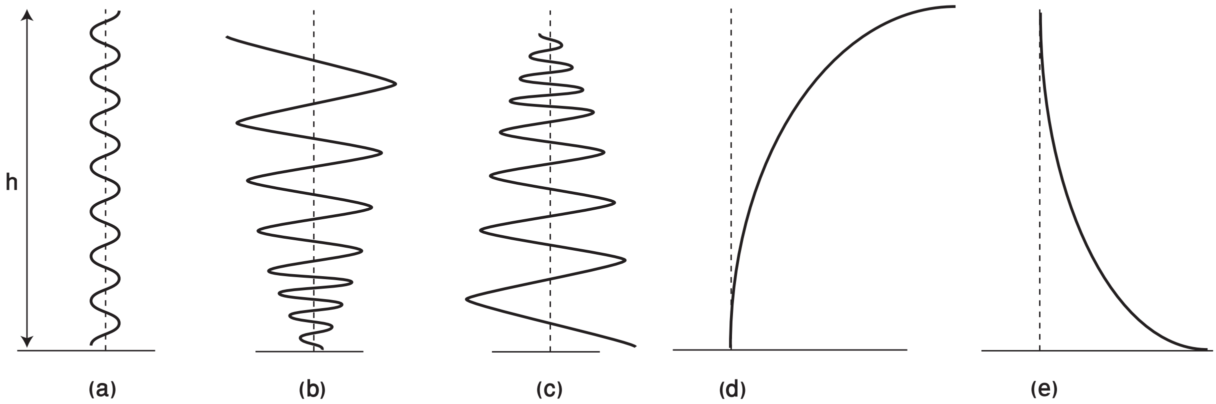
Solution Mid-Term: Modern Physics

1. A heavy atom bounces up and down against the ground and is trapped within a height h above the ground. It possesses gravitational potential energy $V_0 = mgh$. Which of the following diagrams represents a plausible sketch of the atom's wavefunction (only real part is drawn)? The dashed line shows the zero of the wavefunction.



Answer 1:

Correct answer is (b).

As the wave number k depends upon the difference $\sqrt{E - V_0}$, the wavelength λ decreases with increase in k. As V_0 increases with the increase in height, so k decreases with height, hence λ would increase with height.

2. The levels inside an infinite well of length L are quantized according to,

$$E_n = n^2 \frac{\hbar^2 \pi^2}{2mL^2}.$$

What is the maximum wavelength of light that can be absorbed by the system?

(a) $\frac{4mL^2c}{3\pi\hbar}$

(b) $\frac{16\pi mL^2c}{\hbar}$

(c) $\frac{\hbar^2\pi^2}{2mL^2}$

(d) The maximum wavelength is infinity.

(e) none of these answers is correct.

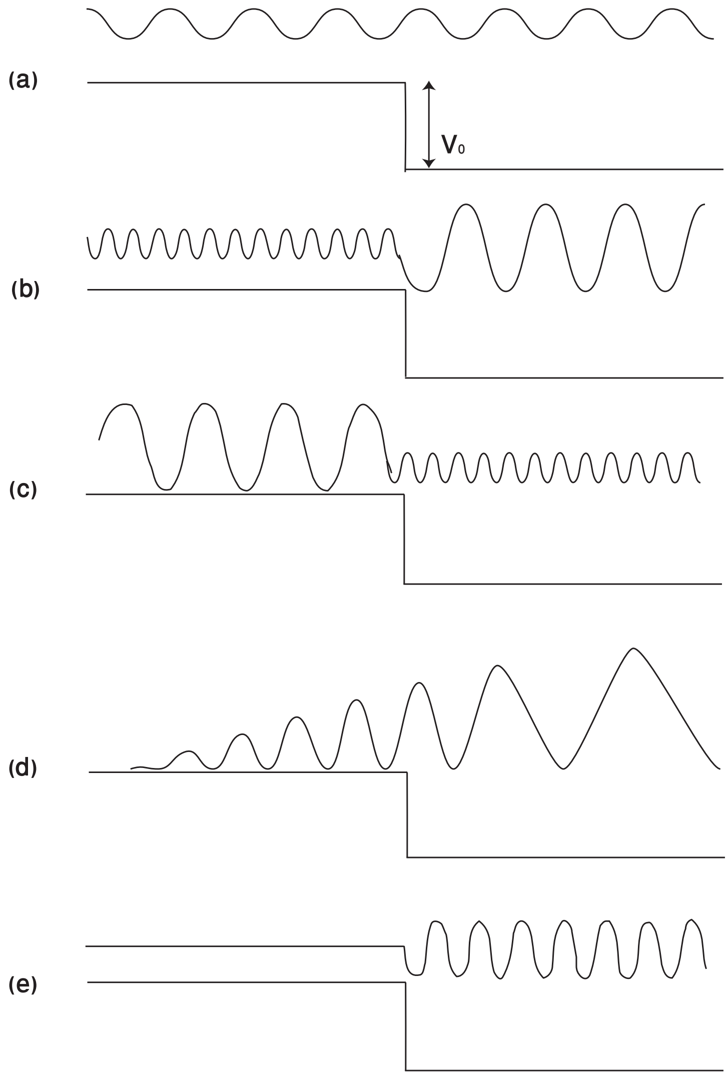
Answer 2:

Correct answer is (a).

Wave length is inversely proportional to energy gaps. It means that for maximum wave length, the energy gap should be minimum. Hence, we need to consider the two lowest possible energy levels. Their gap is,

$$\begin{aligned}
E_2 - E_1 &= 2^2\frac{\pi^2\hbar^2}{2mL^2} - 1^2\frac{\pi^2\hbar^2}{2mL^2} \\
\Delta E &= (4-1)\frac{\pi^2\hbar^2}{2mL^2} \\
\Rightarrow \quad \frac{hc}{\lambda} &= \frac{3\pi^2\hbar^2}{2mL^2} \\
\frac{2\pi\hbar c}{\lambda} &= \frac{3\pi^2\hbar^2}{2mL^2} \qquad \Rightarrow \qquad \lambda = \frac{4mL^2c}{3\pi\hbar}
\end{aligned}$$

3. Which of the following is a reasonable sketch of the real part of the wavefunction where a particle encounters a potential drop.?



Answer 3:

As $k \propto \sqrt{E-V}$, we can say that $\lambda \propto \dfrac{1}{\sqrt{E-V}}$. It is evident from Fig. (c) that it is the correct choice.

4. An electron in an infinite well has energy 1eV, while the ground state energy is 1 μeV. ($1\mu\text{eV} = 10^{-6}$ eV). What is the approximate probability of locating the particle in the middle half of the well?

   (a) $\dfrac{1}{4}$

   (b) $\dfrac{1}{2}$

   (c) $\dfrac{1}{\sqrt{2}}$

   (d) 1.

(e) This is a meaningless question.

Answer 4:

Correct answer is (b).

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} = n^2 E_1$$

Now it is given that $E_1 = 10^{-6}$ eV and $E_1 = 1$ eV'

$$\begin{aligned} 1 \text{ eV} &= n^2 \times 10^6 \text{ eV} \\ \Rightarrow \quad n^2 &= 10^6 \quad \Rightarrow \quad n = 10^3 = 1000 \end{aligned}$$

Our task is to find the particle in the middle half of the well. As n is quite large, the wavefunction would be almost uniform. So the probability to find the particle in the middle half would almost be 1/2. Alternatively, one can check this mathematically.

$$\begin{aligned} \psi_n &= \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \\ P(L/4 < x < 3L/4) &= \int_{L/4}^{3L/4} \left|\psi_n(x)\right|^2 `dx \\ &= \frac{2}{L}\int_{L/4}^{3L/4} \sin^2\left(\frac{n\pi x}{L}\right) \\ &= \frac{2}{L}\int_{L/4}^{3L/4} \left(\frac{1-\cos\left(\frac{2n\pi x}{L}\right)}{2}\right) dx \end{aligned}$$

$$\begin{aligned} &= \frac{1}{L}\int_{L/4}^{3L/4} \left(1 - \cos\frac{2n\pi x}{L}\right) dx \\ &= \frac{1}{L}\left\{ x\Big|_{L/4}^{3L/4} - \frac{\sin\frac{2n\pi x}{L}}{s\frac{2n\pi x}{L}}\Big|_{L/4}^{3L/4} \right\} \\ &= \frac{1}{L}\left\{ \left(\frac{3L}{4} - \frac{L}{4}\right) - \frac{L}{2n\pi}\left(\sin\frac{3n\pi}{2} - \sin\frac{n\pi}{2}\right)\right\} \\ &= \frac{1}{L}\left\{ \left(\frac{L}{2}\right) - \frac{L}{2000\pi}\left(\sin\frac{3000\pi}{2} - \sin\frac{1000\pi}{2}\right)\right\} \quad \because \quad n = 1000 \\ &= \frac{1}{L}\left\{ \left(\frac{L}{2}\right) - \frac{L}{2000\pi}\left(\sin 1500\pi - \sin 500\pi\right)\right\} \\ &= \frac{1}{L}\left\{ \left(\frac{L}{2}\right) - 0\right\} \quad \because \quad \sin(n\pi) = 0 \text{ where } n = 0, 1, 2, 3\ldots \\ &= \frac{1}{2}. \end{aligned}$$

5. A free electron of energy E has a de Broglie wavelength $\lambda = h/p = h/\sqrt{2mE}$ and speed v. In the presence of an electric field, it acquires a potential energy $V(x)$. Hence the total energy changes, and the speed of the electron changes to v'. What is the value of refractive index $n = v/v'$? (Assume $E > V(x)$.)

   (a) $\sqrt{\dfrac{E}{E - V(x)}}$

   (b) 1 (one)

   (c) $\sqrt{\dfrac{E}{V(x)}}$

   (d) $\sqrt{\dfrac{E - V(x)}{V(x)}}$

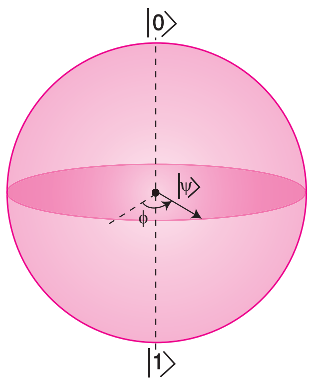
   (e) $\sqrt{\dfrac{E - V(x)}{E}}$

   Answer 5:

   Correct answer is (a).

$$
\begin{aligned}
p &= \sqrt{2mE} = mv \\
\Rightarrow \quad v &= \frac{p}{m} = \frac{\sqrt{2mE}}{m} \\
v &= \sqrt{\frac{2E}{m}} \\
n &= \frac{v}{v'} = \frac{\sqrt{2E/m}}{\sqrt{2(E - V(x))/m}} = \sqrt{\frac{E}{E - V(x)}}.
\end{aligned}
$$

6. A qubit is in the state $|\psi\rangle = \dfrac{1}{\sqrt{\sqrt{2}}}\left(\left(\dfrac{1+i}{\sqrt{2}}\right)|0\rangle + |1\rangle\right)$. On the Bloch sphere, this state is represented by a vector whose tip lies on the equatorial plane. What is its azimuthal angle ϕ?



(a) $\phi = \pi$

(b) $\phi = 0$

(c) $\phi = -\frac{\pi}{4}$

(d) $\phi = \frac{\pi}{4}$

(e) It could be $\phi = \frac{\pi}{4}$ or $\phi = -\frac{\pi}{4}$.

Answer 6:

We will not grade this because of the typo. Nevertheless, here is how you solve it.

Correct answer is (c).

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\left(\frac{1+i}{\sqrt{2}}\right)|0\rangle + |1\rangle\right)$$

Compare $\frac{1+i}{\sqrt{2}}$ with $a+ib$,

$$\begin{aligned}
\left|\frac{1+i}{\sqrt{2}}\right| &= \left|\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right| \\
\gamma &= \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{\frac{1}{2}+\frac{1}{2}} = 1 \\
\theta &= \tan^{-1}\left(\frac{1}{\sqrt{2}}/\frac{1}{\sqrt{2}}\right) = \tan^{-1}(1) = \frac{\pi}{4} \\
\Rightarrow \qquad \frac{1+i}{\sqrt{2}} &= e^{i\pi/4}
\end{aligned}$$

Now $|\psi\rangle$ can be written as,

$$\begin{aligned}
|\psi\rangle &= \frac{1}{\sqrt{2}}\left(e^{i\pi/4}|0\rangle + |1\rangle\right) \\
&= \frac{e^{i\pi/4}}{\sqrt{2}}\left(|0\rangle + e^{-i\pi/4}|1\rangle\right)
\end{aligned}$$

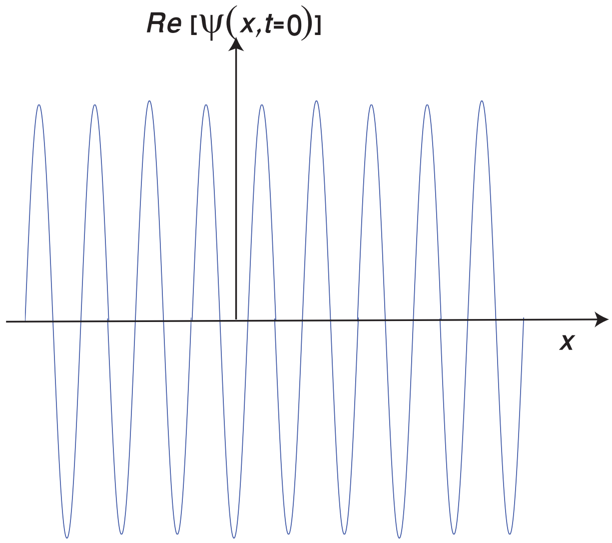
By comparing it with general form,

$$|\psi\rangle = \sin\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\cos\left(\frac{\theta}{2}\right)|1\rangle$$

We get $\phi = -\pi/4$. The upfront factor $e^{i\pi/4}$ is a global phase and inconsequential.

7. A particle is described by the wavefunction $\Psi(x,t) = e^{i(kx-\omega t)}$ and can be thought of a plane wave traveling along the x axis. The real part at $t=0$ is shown in the

accompanying diagram. (The wavefunction extends from $-\infty$ to ∞ which of course we cannot show on paper.) Which of the following statements *most* accurately describes the probability of finding the particle.



(a) It is equally likely to find the particle anywhere along the x axis.

(b) I have no idea how to answer this question.

(c) It is most likely to be found in the peaks of the wave.

(d) It is most likely to be found in the peaks or the troughs the wave.

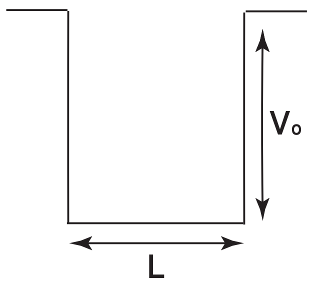
(e) The position of the particle depends on *when* I make a measurement.

Answer 7:

Correct answer is (a).

As $p = \hbar k$ and k is defined for a free particle, so it's momentum is defined. Hence Δx is infinity. So, it is equally likely to locate particle anywhere. Alternatively, $\psi = Ae^{i(kx-\omega t)}$ for a traveling wave. The probability density $|\psi|^2$ is independent of x.

8. An electron is trapped in a quantum dot of diameter L. The electron is in a potential well of depth V_0.



The energy values are approximately the same as the infinite well,

$$E = n^2 \frac{\pi^2 \hbar^2}{2mL^2}. \tag{1}$$

A laser photon of energy E_{photon} shines on the quantum dot in the ground state. What should be the minimum diameter if the electron is to always remain confined in the quantum dot? The dot absorbs the energy E_{photon}.

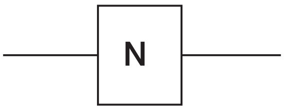
Answer 8:

If the electron is to remained confined, the energy after absorption must be smaller that V_0. Thus

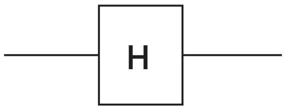
$$\begin{aligned} E_n + E_{\text{photon}} &< V_0 \\ n^2 \frac{\pi^2 \hbar^2}{2mL^2} &< V_0 - E_{\text{photon}} \\ \frac{2mL^2}{n^2 \pi^2 \hbar^2} &> \frac{1}{V_0 - E_{\text{photon}}} \\ L^2 &> n^2 \frac{\pi^2 \hbar^2}{2m(V_0 - E_{\text{photon}})} \\ L &> \frac{\pi \hbar}{\sqrt{2m(V_0 - E_{\text{photon}})}} . \end{aligned}$$

9. Can you draw a quantum circuit (explaining how it works) that generates the three qubit entangled state $\frac{1}{\sqrt{2}}\big(|0\rangle \, |0\rangle \, |0\rangle + |1\rangle \, |1\rangle \, |1\rangle \big)$?

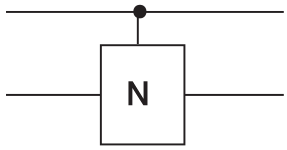
You are allowed to use one or more of the following quantum gates, whose truth tables are provided below.



Input	Output
$\|0\rangle$	$\|1\rangle$
$\|1\rangle$	$\|0\rangle$



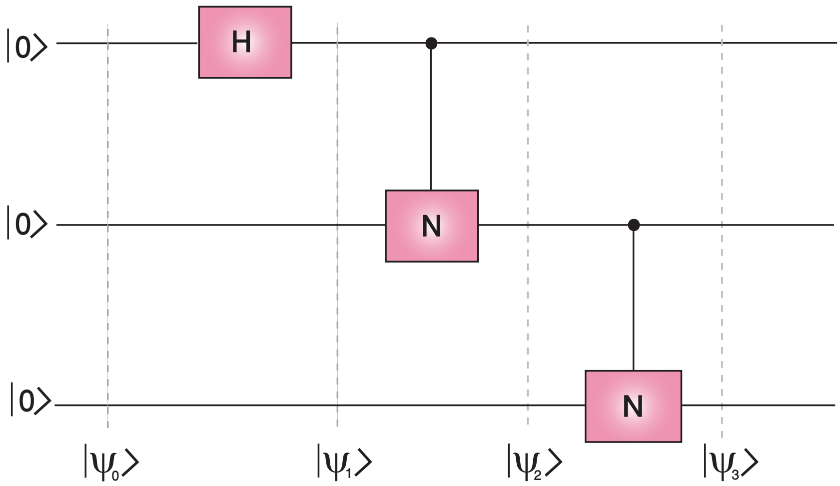
Input	Output
$\|0\rangle$	$\frac{1}{\sqrt{2}}(\|1\rangle+\|0\rangle)$
$\|1\rangle$	$\frac{1}{\sqrt{2}}(\|1\rangle-\|0\rangle)$



Input	Output
$\|0\rangle\|0\rangle$	$\|0\rangle\|0\rangle$
$\|0\rangle\|1\rangle$	$\|0\rangle\|1\rangle$
$\|1\rangle\|0\rangle$	$\|1\rangle\|1\rangle$
$\|1\rangle\|1\rangle$	$\|1\rangle\|0\rangle$

Note that your circuit will use three lines representing three qubits.

Answer 9:



The circuit is shown here. Let's propagate the state through the circuit.

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\begin{aligned}
|\psi_0\rangle &= |0\rangle\,|0\rangle\,|0\rangle \\
|\psi_1\rangle &= \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right)|0\rangle\,|0\rangle \\
|\psi_2\rangle &= \left(\frac{|0\rangle\,|0\rangle + |1\rangle\,|1\rangle}{\sqrt{2}}\right)|0\rangle \\
|\psi_3\rangle &= \left(\frac{|0\rangle\,|0\rangle\,|0\rangle + |1\rangle\,|1\rangle\,|1\rangle}{\sqrt{2}}\right)
\end{aligned}
$$