Solution Quiz $2a$: Modern Physics

1. (a) A qubit is described by the state,

$$|\psi_i\rangle = \frac{i}{\sqrt{3}}|\,0\rangle + \sqrt{\frac{2}{3}}|1\rangle.$$

What is the probability that the system is measured in the final state:

$$|\psi_f\rangle = \left(\frac{1+i}{\sqrt{3}}\right)|\,0\rangle + \sqrt{\frac{1}{3}}|1\rangle.$$

(b) Show $|\psi_f\rangle$ on Bloch sphere.

Answer 1

(a) In order to find the probability of measuring the state in $|\psi_f\rangle$, first we need to find the overlap of $|\psi_f\rangle$ with $|\psi_i\rangle$. Note that writing the bra involves taking the complex conjugate of the coefficients.

$$\langle\psi_f|\psi_i\rangle = \left(\frac{1-i}{\sqrt{3}}\langle 0| + \frac{1}{\sqrt{3}}\langle 1|\right)\left(\frac{i}{\sqrt{3}}|\,0\rangle + \sqrt{\frac{2}{3}}|1\rangle\right)$$

Since the overlap between orthogonal states $\langle 0|$ and $|1\rangle$ is zero, i.e.,

$$\langle 0|1\rangle = \langle 1|\,0\rangle = 0 \quad \text{and} \quad \langle 0|\,0\rangle = \langle 1|\,1\rangle = 1$$

we obtain,

$$\begin{aligned}\langle\psi_f|\psi_i\rangle &= \left(\frac{1-i}{\sqrt{3}}\right)\langle 0|\left(\frac{i}{\sqrt{3}}\right)|\,0\rangle + \left(\sqrt{\frac{1}{3}}\langle 1|\right)\left(\sqrt{\frac{2}{3}}|1\rangle\right)\\ &= \frac{(1-i)i}{3}\langle 0|\,0\rangle + \frac{\sqrt{2}}{3}\langle 1|1\rangle\\ &= \frac{(1+i)}{3} + \frac{\sqrt{2}}{3}.\end{aligned}$$

To find the probability we need to take the modulus square of the overlap.

$$\begin{aligned}\Big|\langle\psi_f|\psi_i\rangle\Big|^2 &= \left|\frac{i+1+\sqrt{2}}{3}\right|^2\\ &= \left(\frac{-i+1+\sqrt{2}}{3}\right)\left(\frac{i+1+\sqrt{2}}{3}\right)\\ &= \frac{4+2\sqrt{2}}{9} \approx 0.76\ \cdot\end{aligned}$$

(b) The general form of state being represented on Bloch sphere is,

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|\,0\rangle \; + \; e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\,1\rangle$$

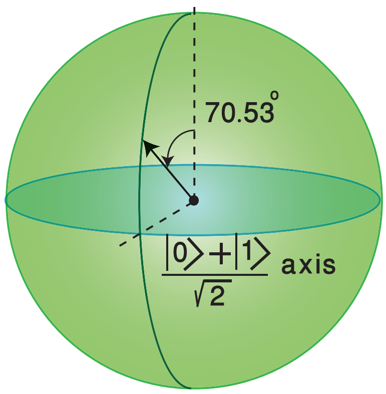
Now compare this with $|\psi_f\rangle$ and use this comparison to determine θ and ϕ. These angles will help us place the state on the Bloch sphere.

$$\begin{aligned}
|\psi_f\rangle &= \left(\frac{1+i}{\sqrt{3}}|\,0\rangle + \frac{1}{\sqrt{3}}|1\rangle\right) \\
\Rightarrow \qquad \cos^2\left(\frac{\theta}{2}\right) &= \left|\frac{1+i}{\sqrt{3}}\right|^2 = \left(\frac{1-i}{\sqrt{3}}\right)\left(\frac{1+i}{\sqrt{3}}\right) = \frac{1^2-i^2}{3} = \frac{2}{3} \\
\cos\frac{\theta}{2} &= \sqrt{\frac{2}{3}} \\
\theta &= 2\cos^{-1}\left(\sqrt{\frac{2}{3}}\right) = 70.53^\circ.
\end{aligned}$$

Now we can verify whether this is the correct θ by plugging it in $\sin\left(\frac{\theta}{2}\right)$.

$$\sin\left(\frac{70.53^\circ}{2}\right) = \sqrt{\frac{1}{3}}$$

which is the correct expression for the coefficient with $|1\rangle$. Furthermore direct comparison between the state $|\psi_f\rangle$ and the general form yields $\phi = 0$.



2. A state is represented by the wave function

$$|\psi\rangle = a|\,0\rangle + e^{i\phi}b|1\rangle,$$

where a and b are real numbers. An experiment finds the probabilities of obtaining

various states in the experiment on multiple copies.

$$\text{Prob. (obtaining } |\,0\rangle) = \frac{1}{2}$$
$$\text{Prob. (obtaining } |\,1\rangle) = \frac{1}{2}$$
$$\text{Prob. }\left(\text{obtaining } \frac{|\,0\rangle + |1\rangle}{\sqrt{2}}\right) = \frac{3}{4}.$$

Estimate a, b and ϕ.

Answer 2:

We know that Prob.(obtaining $|\,0\rangle$ is given by,

$$|a|^2 = \frac{1}{2} \quad \Rightarrow \quad a = \frac{1}{\sqrt{2}} \quad (\because\ a \text{ is real and positive})$$

Similarly Prob.(obtaining $|\,1\rangle$ is given by,

$$\begin{aligned}
\text{Prob.(obtaining } |\,1\rangle) &= |e^{i\phi}\, b|^2 \\
&= e^{-i\phi}\, b^*\, e^{i\phi}\, b \\
&= |b|^2 \\
\Rightarrow \quad |b|^2 &= \frac{1}{2} \quad \Rightarrow \quad b = \frac{1}{\sqrt{2}}\ .
\end{aligned}$$

We have found a and b. The remaining task is to determine ϕ. For this we need to measure along another axis.

The Prob.$\left(\text{obtaining } \frac{|\,0\rangle + |1\rangle}{\sqrt{2}}\right)$ is given by,

$$\left|\left(\frac{\langle 0| + \langle 1|}{\sqrt{2}}\right)\left(\frac{|\,0\rangle + e^{i\phi}|1\rangle}{\sqrt{2}}\right)\right|^2 = \frac{3}{4}$$

Cross terms vanish resulting in,

$$\begin{aligned}
\left|\frac{\langle 0|\,0\rangle + e^{i\phi}\langle 1|1\rangle}{2}\right|^2 &= \left|\frac{1 + e^{i\phi}}{2}\right|^2 \\
\left(\frac{1 + e^{-i\phi}}{2}\right)\left(\frac{1 + e^{i\phi}}{2}\right) &= \frac{3}{4}
\end{aligned}$$

yielding,

$$\begin{aligned}
1 + e^{-i\phi} + e^{i\phi} + 1 &= 3 \\
\Rightarrow \qquad 2 + 2\cos\phi &= 3 \\
2\cos\phi &= 1 \\
\cos\phi &= \frac{1}{2} \\
\phi &= \cos^{-1}\left(\frac{1}{2}\right) \\
&= \pm 60^\circ.
\end{aligned}$$

Hence ϕ could be 60° or -60°. We could in fact determine the angle if we had the probability results when measured in some other bases, such as probability of obtaining $\dfrac{|0\rangle + i|1\rangle}{\sqrt{2}}$.