Solution Quiz $2b$: Modern Physics

1. (a) A qubit is described by the state,

$$|\psi_i\rangle = \frac{1}{2}|\,0\rangle + i\frac{\sqrt{3}}{2}|1\rangle.$$

What is the probability that the system is measured in the final state:

$$|\psi_f\rangle = \left(\frac{1+i}{\sqrt{3}}\right)|0\rangle + \sqrt{\frac{1}{3}}|1\rangle.$$

(b) Show $|\psi_f\rangle$ on Bloch sphere.

Answer 1:

(a) In order to find the probability of measuring the state in $|\psi_f\rangle$, first we need to find the overlap of $|\psi_f\rangle$ with $|\psi_i\rangle$. Note that writing the bra involves taking the complex conjugate of the coefficients.

$$\langle\psi_f|\psi_i\rangle = \left(\frac{1-i}{\sqrt{3}}\langle 0| + \frac{1}{\sqrt{3}}\langle 1|\right)\left(\frac{1}{2}|\,0\rangle + \frac{i\sqrt{3}}{2}|1\rangle\right)$$

Now as we know the overlap between orthogonal states $\langle 0|$ and $|1\rangle$ is zero,i.e.,

$$\langle 0|1\rangle = \langle 1|\,0\rangle = 0 \quad \text{and} \quad \langle 0|\,0\rangle = \langle 1|\,1\rangle = 1$$

we obtain,

$$\begin{aligned}
\langle\psi_f|\psi_i\rangle &= \left(\frac{1-i}{\sqrt{3}}\langle 0|\right)\left(\frac{1}{2}|\,0\rangle\right) + \left(\sqrt{\frac{1}{3}}\langle 1|\right)\left(\frac{i\sqrt{3}}{2}|1\rangle\right) \\
&= \left(\frac{1-i}{\sqrt{3}}\right)\left(\frac{1}{2}\right)\langle 0|\,0\rangle + \left(\sqrt{\frac{1}{3}}\right)\left(\frac{i\sqrt{3}}{2}\right)\langle 1|1\rangle \\
\langle\psi_f|\psi_i\rangle &= \frac{(1-i)}{2\sqrt{3}} + \frac{i\sqrt{3}}{2\sqrt{3}}
\end{aligned}$$

To find the probability we need to take the modulus square of the overlap.

$$\begin{aligned}
\Big|\langle\psi_f|\psi_i\rangle\Big|^2 &= \left|\frac{1-i+i\sqrt{3}}{2\sqrt{3}}\right|^2 \\
&= \left|\left(\frac{1-i(1-\sqrt{3})}{2\sqrt{3}}\right)\right|^2 \\
&= \left(\frac{1-i(1-\sqrt{3})}{2\sqrt{3}}\right)\left(\frac{1+i(1-\sqrt{3})}{2\sqrt{3}}\right) \\
&= \frac{5-2\sqrt{3}}{12} \approx 0.12\cdot
\end{aligned}$$

(b) The general form of state being represented on Bloch sphere is,

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|\,0\rangle \;+\; e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\,1\rangle$$

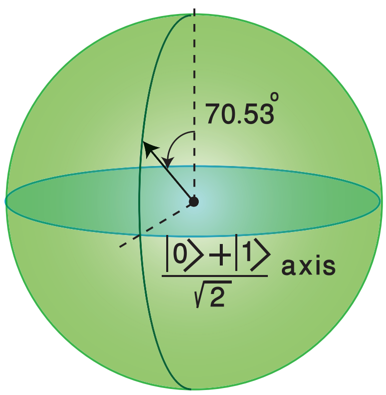
Now compare this with $|\psi_f\rangle$ and use this comparison to determine θ and ϕ. These angles will help us place the state on the Bloch sphere.

$$\begin{aligned}
|\psi_f\rangle &= \left(\frac{1+i}{\sqrt{3}}|\,0\rangle + \frac{1}{\sqrt{3}}|1\rangle\right) \\
\Rightarrow \qquad \cos^2\left(\frac{\theta}{2}\right) &= \left|\frac{1+i}{\sqrt{3}}\right|^2 = \left(\frac{1-i}{\sqrt{3}}\right)\left(\frac{1+i}{\sqrt{3}}\right) = \frac{1^2-i^2}{3} = \frac{2}{3} \\
\cos\frac{\theta}{2} &= \sqrt{\frac{2}{3}} \\
\theta &= 2\cos^{-1}\left(\sqrt{\frac{2}{3}}\right) = 70.53^\circ
\end{aligned}$$

Now we can verify whether this is the correct θ by plugging it in $\sin\left(\frac{\theta}{2}\right)$.

$$\sin\left(\frac{70.53^\circ}{2}\right) = \sqrt{\frac{1}{3}}$$

which is the correct expression for the coefficient with $|1\rangle$. Furthermore direct comparison between the state $|\psi_f\rangle$ and the general form yields $\phi = 0$.



2. A state is represented by the wave function

$$|\psi\rangle = \alpha|\,0\rangle + e^{i\phi}\beta|1\rangle,$$

where α and β are real numbers. An experiment finds the probabilities of obtaining

various states in the experiment on multiple copies.

$$\text{Prob (obtaining } |\,0\rangle) = \frac{1}{3}$$
$$\text{Prob (obtaining } |\,1\rangle) = \frac{2}{3}$$
$$\text{Prob}\left(\text{obtaining } \frac{1}{\sqrt{3}}|\,0\rangle + \sqrt{\frac{2}{3}}|1\rangle\right) = \frac{1}{3}.$$

Estimate α, β and ϕ.

<u>Answer 2:</u>

We know that Prob.(obtaining $|\,0\rangle$ is given by,

$$|\alpha|^2 = \frac{1}{3} \quad \Rightarrow \quad \alpha = \frac{1}{\sqrt{3}} \quad (\because\ \alpha \text{ is real and positive})$$

Similarly Prob.(obtaining $|\,1\rangle$ is given by,

$$\begin{aligned}\text{Prob.(obtaining } |\,1\rangle) &= |e^{i\phi}\ \beta|^2 \\ &= e^{-i\phi}\ \beta^*\ e^{i\phi}\ \beta \\ &= |\beta|^2 \\ \Rightarrow \quad |\beta|^2 &= \frac{2}{3} \quad \Rightarrow \quad \beta = \sqrt{\frac{2}{3}}\ .\end{aligned}$$

We have found α and β. The remaining task is to determine ϕ. For this we need to measure along another axis. The Prob.$\left(\text{obtaining } \frac{|\,0\rangle + \sqrt{2}|1\rangle}{\sqrt{3}}\right)$ is given by,

$$\left|\left(\frac{\langle 0| + \sqrt{2}\langle 1|}{\sqrt{3}}\right)\left(\frac{|\,0\rangle + e^{i\phi}\sqrt{2}|1\rangle}{\sqrt{3}}\right)\right|^2 = \frac{1}{3}$$

Cross terms vanish resulting in,

$$\begin{aligned}\left|\frac{\langle 0|\,0\rangle + 2e^{i\phi}\langle 1|1\rangle}{3}\right|^2 &= \left|\frac{1 + 2e^{i\phi}}{3}\right|^2 \\ \left(\frac{1 + 2e^{-i\phi}}{3}\right)\left(\frac{1 + 2e^{+i\phi}}{3}\right) &= \frac{1}{3}\end{aligned}$$

yielding,

$$\begin{aligned}
1 + 2e^{-i\phi} + 2e^{i\phi} + 4 &= 3 \\
5 + 2(e^{i\phi} + e^{-i\phi}) &= 3 \\
\Rightarrow \qquad 5 + 2(2\cos\phi) &= 3 \\
2\cos\phi &= -1 \\
\cos\phi &= \frac{-1}{2} \\
\phi &= \cos^{-1}\left(\frac{-1}{2}\right) \\
&= \pm 120^\circ.
\end{aligned}$$

Hence ϕ could be $+120^\circ$ or -120°. We could in fact determine the angle if we had the probability results when measured in some other bases, such as probability of obtaining $\frac{|0\rangle + i\sqrt{2}|1\rangle}{\sqrt{3}}$.