Solution Quiz $3a$: Modern Physics

Date: 12 **April** 2018

Useful Formulae

$$h = 6.63 \times 10^{-34} \text{ Js} \qquad m_e = 9.11 \times 10^{-31} \text{ kg}$$

$$E_n = \frac{-13.6}{n^2} \text{ eV in a } H \text{ atom} \qquad 1 \text{ eV} = 1.6 \times 10^{-19} \text{ J.}$$

1. The radial wavefunction for $n = 2$, $\ell = 1$ is given by,

$$R(r) = A\left(\frac{r}{a_0}\right)e^{-(r/2a_0)},$$

where A is a constant and a_0 is the Bohr's radius. Where is the maximum likelihood of finding the electron in this state? (5 marks)

Answer 1

Radial probability density $p_r(r)$ is

$$\begin{aligned} p_r(r) &= 4\pi r^2 |R(r)|^2 \\ &= \frac{4\pi A^2}{a_0^2} r^4 e^{-r/a_0} \end{aligned}$$

For finding the r at which this density is maximum,

$$\begin{aligned} \frac{dp_r(r)}{dr} &= \frac{4\pi A^2}{a_0^2}(4r^3\ e^{-r/a_0} - \frac{1}{a_0} r^4\ e^{-r/a_0}) = 0 \\ \Rightarrow \quad 4r^3 - \frac{1}{a_0} r^4 &= 0 \\ \Rightarrow \quad r &= 4a_0. \end{aligned}$$

The maximum likelihood is at $r = 4a_0$.

2. A particle of mass m is at a fixed radius R from the origin. The moment of inertia is $I = mR^2$. The time dependent Schrodinger equation is,

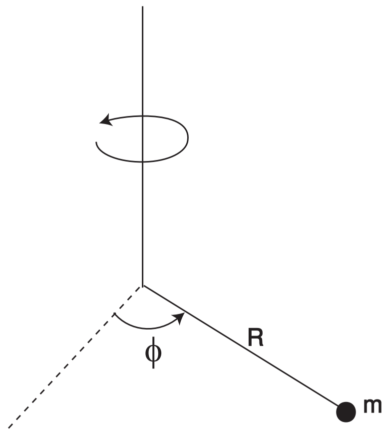
$$-\frac{\hbar^2}{2I}\frac{\partial^2\Psi(\phi,t)}{\partial\phi^2} = i\hbar\frac{\partial\Psi(\phi,t)}{\partial t},$$

where ϕ is the variable angle in space. Use separation of variable to

(a) write down the spatial (space) and temporal(time) component of the schrodinger equation. (5 marks)

(b) Solve the spatial part to find $\Phi(\phi)$. (5 marks)

(c) We want the spatial part $\Phi(\phi)$ to be single-valued meaning that if ϕ changes by 2π or multiples thereof, the function does not change, i.e., $\Phi(\phi+2\pi n) = \Phi(\phi)$.What kind of quantization does this led to? Comment. (5 marks)



Answer 2

(a)

$$\Psi(\phi,t) = \Phi(\phi)\ f(t)$$

Inserting into the L.H.S. and R.H.S.

$$-\frac{\hbar^2}{2I}\frac{\partial^2}{\partial\phi^2}\big(\Phi(\phi)\ f(t)\big) = i\hbar\frac{\partial}{\partial t}\big(\Phi(\phi)\ f(t)\big)$$
$$-\frac{\hbar^2}{2I}\left(\frac{\partial^2\Phi(\phi)}{\partial\phi^2}\right)f(t) = i\hbar\ \Phi(\phi)\frac{\partial}{\partial t}f(t)$$

Divide by $\Phi(\phi)\ f(t)$:

$$-\frac{\hbar^2}{2I}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}\frac{1}{\Phi(\phi)} = i\hbar\ \frac{1}{f(t)}\ \frac{\partial}{\partial t}f(t)$$

Each side is equal to a constant, say E. Hence spatial part is

$$\frac{d^2\Phi(\phi)}{d\phi^2} + \frac{2IE}{\hbar^2}\Phi(\phi) = 0$$

and the temporal part is

$$\frac{\partial}{\partial t} f(t) = -\frac{i}{\hbar}\, E f(t).$$

(b) The spatial part is:

$$\frac{d^2\Phi(\phi)}{d\phi^2} + \frac{2IE}{\hbar^2}\Phi(\phi) = 0$$

The auxiliary equation is:

$$s^2 + \left(\frac{2IE}{\hbar^2}\right) = 0$$
$$s = \pm\frac{i\sqrt{2IE}}{\hbar}$$

Hence the solution is:

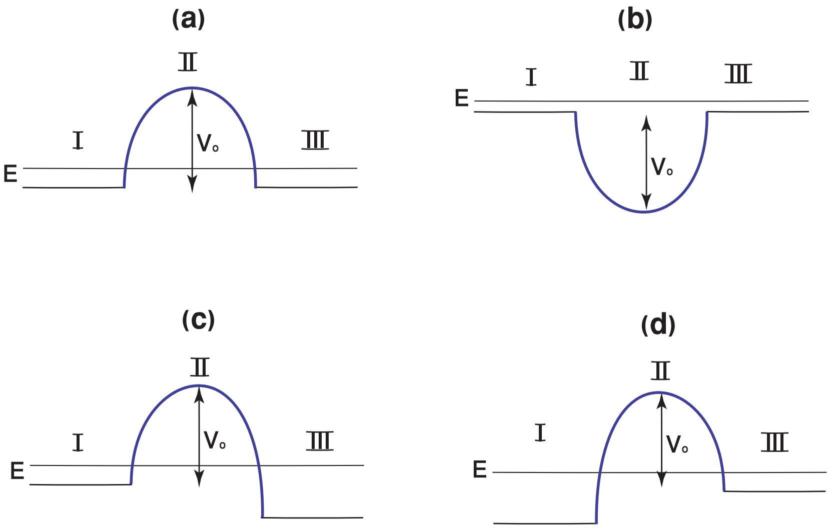
$$\Phi(\phi) = Ae^{+i\frac{\sqrt{2IE}}{\hbar}\phi} + Be^{-i\frac{\sqrt{2IE}}{\hbar}\phi}. \tag{1}$$

(c) The boundary condition is:

$$\begin{aligned}\Phi(\phi + 2\pi n) &= \Phi(\phi)\\ \Phi(\phi + 2\pi n) &= Ae^{+i\frac{\sqrt{2IE}}{\hbar}(\phi+2\pi n)} + Be^{-i\frac{\sqrt{2IE}}{\hbar}(\phi+2\pi n)}\\ &= Ae^{+i\frac{\sqrt{2IE}}{\hbar}\phi}e^{i\frac{\sqrt{2IE}}{\hbar}2\pi n} + Be^{-i\frac{\sqrt{2IE}}{\hbar}\phi}e^{-i\frac{\sqrt{2IE}}{\hbar}2\pi n} \end{aligned} \tag{2}$$

Equations (1) and (2) are equal requiring that $2\pi\frac{\sqrt{2IE}}{\hbar}n$ is an integral multiple of 2π since $e^{\pm i2\pi p} = 1$ where p is an integer. Hence we require that $\frac{\sqrt{2IE}}{\hbar}$ is an integer, which is the required quantization condition.

3. An electron is injected into a potential energy landscape from the left region I as shown below. It encounters a potential step. The energy of the electron is E and $E < |V_0|$. If the electron is to emerge in region III with a faster speed, the appropriate potential step is given by which of the following?



(e) The speed of the electron cannot increase.

Answer 3

In region III, we need v (speed) to be higher, so momentum ($p = \hbar k$) needs to be higher. A higher k is a shorter wavelength so, $\sqrt{E - V}$ should be higher in region III than in region I. This is the scenario in option (c).