44

Formal development of fermionic path integrals

Prerequisite: 43

In section 43, we formally defined the fermionic path integral for a free Dirac field Ψ via

$$Z_0(\overline{\eta},\eta) = \int \mathcal{D}\Psi\,\mathcal{D}\overline{\Psi}\,\exp\left[i\int d^4x\,\overline{\Psi}(i\partial\!\!\!/ - m)\Psi + \overline{\eta}\Psi + \overline{\Psi}\eta\right]$$

$$= \exp\left[i\int d^4x\,d^4y\,\overline{\eta}(x)S(x-y)\eta(y)\right], \tag{44.1}$$

where the Feynman propagator $S(x-y)$ is the inverse of the Dirac wave operator:

$$(-i\partial\!\!\!/_x + m)S(x-y) = \delta^4(x-y)\,. \tag{44.2}$$

We would like to find a mathematical framework that allows us to derive this formula, rather than postulating it by analogy.

Consider a set of *anticommuting numbers* or *Grassmann variables* ψ_i that obey

$$\{\psi_i,\psi_j\} = 0\,, \tag{44.3}$$

where $i = 1,\ldots,n$. Let us begin with the very simplest case of $n = 1$, and thus a single anticommuting number ψ that obeys $\psi^2 = 0$. We can define a function $f(\psi)$ of such an object via a Taylor expansion; because $\psi^2 = 0$, this expansion ends with the second term:

$$f(\psi) = a + \psi b\,. \tag{44.4}$$

The reason for writing the coefficient b to the right of the variable ψ will become clear in a moment.

Next we would like to define the derivative of $f(\psi)$ with respect to ψ. Before we can do so, we must decide if $f(\psi)$ itself is to be commuting or anticommuting; generally we will be interested in functions that are themselves commuting. In this case, a in eq. (44.4) should be treated as an ordinary commuting number, but b should be treated as an anticommuting number: $\{b, b\} = \{b, \psi\} = 0$. In this case, $f(\psi) = a + \psi b = a - b\psi$.

Now we can define two kinds of derivatives. The *left derivative* of $f(\psi)$ with respect to ψ is given by the coefficient of ψ when $f(\psi)$ is written with the ψ always on the far left:

$$\partial_\psi f(\psi) = +b\,. \tag{44.5}$$

Similarly, the *right derivative* of $f(\psi)$ with respect to ψ is given by the coefficient of ψ when $f(\psi)$ is written with the ψ always on the far right:

$$f(\psi)\overleftarrow{\partial}_\psi = -b\,. \tag{44.6}$$

Generally, when we write a derivative with respect to a Grassmann variable, we mean the left derivative. However, in section 37, when we wrote the canonical momentum for a fermionic field ψ as $\pi = \partial\mathcal{L}/\partial(\partial_0\psi)$, we actually meant the right derivative. (This is a standard, though rarely stated, convention.) Correspondingly, we wrote the hamiltonian density as $\mathcal{H} = \pi\partial_0\psi - \mathcal{L}$, with $\partial_0\psi$ to the right of π.

Finally, we would like to define a definite integral, analogous to integrating a real variable x from minus to plus infinity. The key features of such an integral over x (when it converges) are linearity,

$$\int_{-\infty}^{+\infty} dx\, cf(x) = c\int_{-\infty}^{+\infty} dx\, f(x)\,, \tag{44.7}$$

and invariance under shifts of the dependent variable x by a constant:

$$\int_{-\infty}^{+\infty} dx\, f(x+a) = \int_{-\infty}^{+\infty} dx\, f(x)\,. \tag{44.8}$$

Up to an overall numerical factor that is the same for every $f(\psi)$, the only possible nontrivial definition of $\int d\psi\, f(\psi)$ that is both linear and shift invariant is

$$\int d\psi\, f(\psi) = b\,. \tag{44.9}$$

Now let us generalize this to $n > 1$. We have

$$f(\psi) = a + \psi_i b_i + \tfrac{1}{2}\psi_{i_1}\psi_{i_2}c_{i_1 i_2} + \ldots + \tfrac{1}{n!}\psi_{i_1}\ldots\psi_{i_n}d_{i_1\ldots i_n}\,, \tag{44.10}$$

where the indices are implicitly summed. Here we have written the coefficients to the right of the variables to facilitate left-differentiation. These coefficients are completely antisymmetric on exchange of any two indices. The left derivative of $f(\psi)$ with respect to ψ_j is

$$\frac{\partial}{\partial\psi_j} f(\psi) = b_j + \psi_i c_{ji} + \ldots + \tfrac{1}{(n-1)!}\psi_{i_2}\ldots\psi_{i_n} d_{j i_2 \ldots i_n} \, . \tag{44.11}$$

Next we would like to find a linear, shift-invariant definition of the integral of $f(\psi)$. Note that the antisymmetry of the coefficients implies that

$$d_{i_1\ldots i_n} = d\,\varepsilon_{i_1\ldots i_n} \, , \tag{44.12}$$

where d is just a number (ordinary if f is commuting and n is even, Grassmann if f is commuting and n is odd, etc.), and $\varepsilon_{i_1\ldots i_n}$ is the completely antisymmetric Levi-Civita symbol with $\varepsilon_{1\ldots n} = +1$. This number d is a candidate (in fact, up to an overall numerical factor, the only candidate!) for the integral of $f(\psi)$:

$$\int d^n\psi\, f(\psi) = d \, . \tag{44.13}$$

Although eq. (44.13) really tells us everything we need to know about $\int d^n\psi$, we can, if we like, write $d^n\psi = d\psi_n \ldots d\psi_1$ (note the backwards ordering), and treat the individual differentials as anticommuting: $\{d\psi_i, d\psi_j\} = 0$, $\{d\psi_i, \psi_j\} = 0$. Then we take $\int d\psi_i = 0$ and $\int d\psi_i\, \psi_j = \delta_{ij}$ as our basic formulae, and use them to derive eq. (44.13).

Let us work out some consequences of eq. (44.13). Consider what happens if we make a linear change of variable,

$$\psi_i = J_{ij}\psi'_j \, , \tag{44.14}$$

where J_{ji} is a matrix of commuting numbers (and therefore can be written on either the left or right of ψ'_j). We now have

$$f(\psi) = a + \ldots + \tfrac{1}{n!}(J_{i_1 j_1}\psi'_{j_1})\ldots(J_{i_n j_n}\psi'_{j_n})\varepsilon_{i_1\ldots i_n} d \, . \tag{44.15}$$

Next we use

$$\varepsilon_{i_1\ldots i_n} J_{i_1 j_1} \ldots J_{i_n j_n} = (\det J)\varepsilon_{j_1\ldots j_n} \, , \tag{44.16}$$

which holds for any $n \times n$ matrix J, to get

$$f(\psi) = a + \ldots + \tfrac{1}{n!}\psi'_{i_1}\ldots\psi'_{i_n}\varepsilon_{i_1\ldots i_n}(\det J) d \, . \tag{44.17}$$

If we now integrate $f(\psi)$ over $d^n\psi'$, eq. (44.13) tells us that the result is $(\det J)d$. Thus,

$$\int d^n\psi\, f(\psi) = (\det J)^{-1} \int d^n\psi'\, f(\psi)\ . \tag{44.18}$$

Recall that, for integrals over commuting real numbers x_i with $x_i = J_{ij}x'_j$, we have instead

$$\int d^n x\, f(x) = (\det J)^{+1} \int d^n x'\, f(x)\ . \tag{44.19}$$

Note the opposite sign on the power of the determinant.

Now consider a quadratic form $\psi^{\mathrm{T}} M\psi = \psi_i M_{ij}\psi_j$, where M is an antisymmetric matrix of commuting numbers (possibly complex). Let us evaluate the gaussian integral $\int d^n\psi\, \exp(\frac{1}{2}\psi^{\mathrm{T}}M\psi)$. For example, for $n = 2$, we have

$$M = \begin{pmatrix} 0 & +m \\ -m & 0 \end{pmatrix}, \tag{44.20}$$

and $\psi^{\mathrm{T}}M\psi = 2m\psi_1\psi_2$. Thus $\exp(\frac{1}{2}\psi^{\mathrm{T}}M\psi) = 1 + m\psi_1\psi_2$, and so

$$\int d^n\psi\, \exp(\tfrac{1}{2}\psi^{\mathrm{T}}M\psi) = m\ . \tag{44.21}$$

For larger n, we use the fact that a complex antisymmetric matrix can be brought to a block-diagonal form via

$$U^{\mathrm{T}}MU = \begin{pmatrix} 0 & +m_1 & & \\ -m_1 & 0 & & \\ & & & \ddots \end{pmatrix}, \tag{44.22}$$

where U is a unitary matrix, and each m_I is real and positive. (If n is odd there is a final row and column of all zeros; from here on, we assume n is even.) We can now let $\psi_i = U_{ij}\psi'_j$; then, dropping the primes, we have

$$\int d^n\psi\, \exp(\tfrac{1}{2}\psi^{\mathrm{T}}M\psi) = (\det U)^{-1}\prod_{I=1}^{n/2}\int d^2\psi_I\, \exp(\tfrac{1}{2}\psi^{\mathrm{T}}M_I\psi)\ , \tag{44.23}$$

where M_I represents one of the 2×2 blocks in eq. (44.22). Each of these two-dimensional integrals can be evaluated using eq. (44.21), and so

$$\int d^n\psi\, \exp(\tfrac{1}{2}\psi^{\mathrm{T}}M\psi) = (\det U)^{-1}\prod_{I=1}^{n/2} m_I\ . \tag{44.24}$$

Taking the determinant of eq. (44.22), we get

$$(\det U)^2(\det M) = \prod_{I=1}^{n/2} m_I^2 \,. \tag{44.25}$$

We can therefore rewrite the right-hand side of eq. (44.24) as

$$\int d^n\psi \,\exp(\tfrac{1}{2}\psi^{\mathrm{T}}M\psi) = (\det M)^{1/2} \,. \tag{44.26}$$

In this form, there is a sign ambiguity associated with the square root; it is resolved by eq. (44.24). However, the overall sign (more generally, any overall numerical factor) will never be of concern to us, so we can use eq. (44.26) without worrying about the correct branch of the square root.

It is instructive to compare eq. (44.26) with the corresponding gaussian integral for commuting real numbers,

$$\int d^n x \,\exp(-\tfrac{1}{2}x^{\mathrm{T}}Mx) = (2\pi)^{n/2}(\det M)^{-1/2} \,. \tag{44.27}$$

Here M is a complex symmetric matrix. Again, note the opposite sign on the power of the determinant.

Now let us introduce the notion of *complex* Grassmann variables via

$$\chi \equiv \tfrac{1}{\sqrt{2}}(\psi_1 + i\psi_2) \,,$$

$$\bar{\chi} \equiv \tfrac{1}{\sqrt{2}}(\psi_1 - i\psi_2) \,. \tag{44.28}$$

We can invert this to get

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} \bar{\chi} \\ \chi \end{pmatrix} . \tag{44.29}$$

The determinant of this transformation matrix is $-i$, and so

$$d^2\psi = d\psi_2 d\psi_1 = (-i)^{-1} d\chi\, d\bar{\chi} \,. \tag{44.30}$$

Also, $\psi_1\psi_2 = -i\bar{\chi}\chi$. Thus we have

$$\int d\chi\, d\bar{\chi}\, \bar{\chi}\chi = (-i)(-i)^{-1}\int d\psi_2 d\psi_1\, \psi_1\psi_2 = 1 \,. \tag{44.31}$$

Thus, if we have a function

$$f(\chi,\bar{\chi}) = a + \chi b + \bar{\chi}c + \bar{\chi}\chi d \,, \tag{44.32}$$

its integral is

$$\int d\chi\, d\bar\chi\, f(\chi,\bar\chi) = d\ . \tag{44.33}$$

In particular,

$$\int d\chi\, d\bar\chi\, \exp(m\bar\chi\chi) = m\ . \tag{44.34}$$

Let us now consider n complex Grassmann variables χ_i and their complex conjugates, $\bar\chi_i$. We define

$$d^n\chi\, d^n\bar\chi \equiv d\chi_n d\bar\chi_n \ldots d\chi_1 d\bar\chi_1\ . \tag{44.35}$$

Then under a change of variable, $\chi_i = J_{ij}\chi'_j$ and $\bar\chi_i = K_{ij}\bar\chi'_j$, we have

$$d^n\chi\, d^n\bar\chi = (\det J)^{-1}(\det K)^{-1}\, d^n\chi'\, d^n\bar\chi'\ . \tag{44.36}$$

Note that we need not require $K_{ij} = J^*_{ij}$, because, as far as the integral is concerned, it does not matter whether or not $\bar\chi_i$ is the complex conjugate of χ_i.

We now have enough information to evaluate $\int d^n\chi\, d^n\bar\chi\, \exp(\chi^\dagger M\chi)$, where M is a general complex matrix. We make the change of variable $\chi = U\chi'$ and $\chi^\dagger = \chi'^\dagger V$, where U and V are unitary matrices with the property that VMU is diagonal with positive real entries m_i. Dropping the primes, we get

$$\begin{aligned}\int d^n\chi\, d^n\bar\chi\, \exp(\chi^\dagger M\chi) &= (\det U)^{-1}(\det V)^{-1}\prod_{i=1}^{n}\int d\chi_i d\bar\chi_i\, \exp(m_i\bar\chi_i\chi_i)\\ &= (\det U)^{-1}(\det V)^{-1}\prod_{i=1}^{n} m_i\\ &= \det M\ .\end{aligned} \tag{44.37}$$

This can be compared to the analogous integral for commuting complex variables $z_i = (x_i + iy_i)/\sqrt{2}$ and $\bar z_i = (x_i - iy_i)/\sqrt{2}$, with $d^n z\, d^n\bar z = d^n x\, d^n y$, namely

$$\int d^n z\, d^n\bar z\, \exp(-z^\dagger M z) = (2\pi)^n(\det M)^{-1}\ . \tag{44.38}$$

We can now generalize eqs. (44.26) and (44.37) by shifting the integration variables, and using shift invariance of the integrals. Thus, by making the replacement $\psi \to \psi - M^{-1}\eta$ in eq. (44.26), we get

$$\int d^n\psi\, \exp(\tfrac{1}{2}\psi^{\mathrm T}M\psi + \eta^{\mathrm T}\psi) = (\det M)^{1/2}\exp(\tfrac{1}{2}\eta^{\mathrm T}M^{-1}\eta)\ . \tag{44.39}$$

(In verifying this, remember that M and its inverse are both antisymmetric.) Similarly, by making the replacements $\chi \to \chi - M^{-1}\eta$ and $\chi^\dagger \to \chi^\dagger - \eta^\dagger M^{-1}$ in eq. (44.37), we get

$$\int d^n\chi\, d^n\bar{\chi}\, \exp(\chi^\dagger M\chi + \eta^\dagger\chi + \chi^\dagger\eta) = (\det M)\exp(-\eta^\dagger M^{-1}\eta)\,. \quad (44.40)$$

We can now see that eq. (44.1) is simply a particular case of eq. (44.40), with the index on the complex Grassmann variable generalized to include both the ordinary spin index α and the continuous spacetime argument x of the field $\Psi_\alpha(x)$. Similarly, eq. (43.21) for the path integral for a free Majorana field is simply a particular case of eq. (44.39). In both cases, the determinant factors are constants (that is, independent of the fields and sources) that we simply absorb into the overall normalization of the path integral. We will meet determinants that cannot be so neatly absorbed in sections 53 and 71.