Field Integral for Massive Electromagnetism

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$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 A_\mu A^\mu + J_\mu A^\mu$$

The first term can be remolded,

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= (\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) \\ &= \partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu \end{aligned}$$

Swapping μ and ν in the third and fourth terms yields,

$$F_{\mu\nu}F^{\mu\nu} = 2\partial_\mu A_\nu \partial^\mu A^\nu - 2\partial_\mu A_\nu \partial^\nu A^\mu$$

So

$$\mathscr{L} = -\frac{1}{2}\big(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\big) + \frac{1}{2}m^2 A_\mu A^\mu + J_\mu A^\mu \tag{1}$$

The field integral is

$$Z(J) = \int DA \; e^{i\int_{-\infty}^{\infty} d^4x \; \mathscr{L}(x)} \tag{2}$$

The phase is given by $\int_{-\infty}^{\infty} d^4x \; \mathscr{L}(x)$. We need the integral of all terms in (1) and see if they can be written in the standard format given by,

$$\begin{aligned} &\int D\big(\phi(x)\big) \; e^{-\frac{1}{2}\int d^4x \; d^4y \; f(x) \; A(x,y) \; f(y) + \int dx \; b(x) \; f(x)} \\ &= \frac{B}{\big(Det(A(x,y))\big)^{1/2}} e^{-\frac{1}{2}\int dx \; dy \; b(x) \; \big(A(x,y)\big)^{-1} \; b(y)} \end{aligned} \tag{3}$$

The third and fourth terms in equation (3) are already in the desired form. We need to recast the first and fourth terms. We have for the first term

$$\begin{aligned} i\int d^4x \left(-\frac{1}{2}\right)(\partial_\mu A_\nu \partial^\mu A^\nu) &= -\frac{i}{2}\int_{-\infty}^{\infty} d^4x \; \partial_\mu A_\nu \partial^\mu A^\nu \\ &= -\frac{i}{2}\left[A_\nu \partial^\mu A^\nu \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} d^4x \; A_\nu \partial^2 A^\nu\right] \\ &= \frac{i}{2}\int d^4x \; A_\nu \partial^2 A^\nu \\ &= \frac{i}{2}\int d^4x \; A^\mu \; g_{\mu\nu} \; \partial^2 \; A^\nu \end{aligned} \tag{4}$$

While for the second term,

$$i\int d^4x\left(+\frac{1}{2}\right)(\partial_\mu A_\nu\partial^\nu A^\mu) = \frac{i}{2}\left[A_\nu\partial^\nu A^\mu\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} d^4x\ A_\nu\partial_\mu\partial^\nu A^\mu\right]$$
$$= -\frac{i}{2}\int_{-\infty}^{\infty} d^4x\ A_\nu\partial_\mu\partial^\nu A^\mu$$

Swapping ν in upper and lower positions, i.e. $X^\mu Y_\mu = X_\mu Y^\mu$ gives us

$$i\int d^4x\left(+\frac{1}{2}\right)(\partial_\mu A_\nu\partial^\nu A^\mu) = -\frac{i}{2}\int_{-\infty}^{\infty} d^4x\ A^\nu\partial_\mu\partial_\nu A^\mu \tag{5}$$

Inserting equation (4) and equation (5) into equation (2) yields,

$$Z(J) = \int DAe^{i\int d^4x\left(\frac{1}{2}A^\mu g_{\mu\nu}\partial^2A^\nu - \frac{1}{2}A^\mu\partial_\mu\partial_\nu A^\nu\right)+\frac{m^2}{2}A_\mu A^\mu + J_\mu A^\mu}$$
$$= \int DAe^{\frac{i}{2}\int d^4x(A^\mu)(g_{\mu\nu}\partial^2-\partial_\mu\partial_\nu)A^\nu + m^2A_\mu A^\mu + i\int d^4xJ_\mu A^\mu}$$
$$= \int DAe^{\frac{i}{2}\int d^4x(A^\mu)(g_{\mu\nu}(\partial^2+m^2)-\partial_\mu\partial_\nu)A^\nu + i\int d^4xJ_\mu A^\mu}$$
$$Z(J=0) = \int DAe^{\frac{i}{2}\int d^4x(A^\mu)(g_{\mu\nu}(\partial^2+m^2)-\partial_\mu\partial_\nu)A^\nu}$$
$$Z(J) = \frac{Z(J)}{Z(J=0)}$$
$$= \frac{\int DAe^{\frac{i}{2}\int d^4x\left[(A^\mu)(g_{\mu\nu}(\partial^2+m^2)-\partial_\mu\partial_\nu)A^\nu\right] + i\int d^4xJ_\mu A^\mu}}{\int DAe^{\frac{i}{2}\int d^4x\left[(A^\mu)(g_{\mu\nu}(\partial^2+m^2)-\partial_\mu\partial^\nu)A^\nu\right]}}\ .$$

Now let

$$g_{\mu\nu}(\partial^2+m^2) - \partial_\mu\partial_\nu = K_{\mu\nu} \tag{6}$$
$$Z(J) = \frac{\int DAe^{\frac{i}{2}\int d^4x\ \ A^\mu\ \ K_{\mu\nu}A^\nu + i\int d^4xJ_\mu A^\mu}}{\int DAe^{\frac{i}{2}\int d^4x\ \ A^\mu\ \ K_{\mu\nu}A^\nu}}\ .$$

Applying the standard form from equation (3) gives us:

$$Z(J) = e^{-\frac{i}{2}\int d^4x\ \ d^4y\ \ J^\mu(x)(K_{\mu\nu}(x))^{-1}J^\nu(y)}$$
$$= e^{-\frac{1}{2}\int d^4x\ \ d^4y\ \ J^\mu(x)(iK^{-1}_{\mu\nu}(x))J^\nu(y)} \tag{7}$$

Now

$$KK^{-1} = \delta^{(4)}(x-y)$$

where

$$K = g_{\mu\nu}(\partial^2 + m^2) - \partial_\mu\partial_\nu$$
$$K(iK^{-1}) = \big(g_{\mu\nu}(\partial^2 + m^2) - \partial_\mu\partial^\nu\big)(iK^{-1}) = i\delta^{(4)}(x-y)$$

Now iK^{-1} is the propagator (Green's function) for the system. Let's denote it by $G_{\mu\nu}$. Then we can find this function by re-writing the equation above as:

$$\big(g^{\mu\nu}(\partial^2 + m^2) - \partial^\mu\partial^\nu\big)G_{\nu\lambda} = ig^\mu_\lambda\delta^{(4)}(x-y) \tag{8}$$

In order to solve this equation, let's take its F.T.

$$\big(g^{\mu\nu}(-p^2 + m^2) + p^\mu p^\nu\big)\widetilde{G}_{\nu\lambda}(p) = ig^\mu_\lambda \tag{9}$$

The "+" sign before $p^\mu p^\nu$ stems from $(-ip^\mu)(ip^\nu) = +p^\mu p^\nu$. Its solution is given by

$$\widetilde{G}_{\nu\lambda}(p) = \frac{-i(g_{\nu\lambda} - p_\nu p_\lambda/m^2)}{p^2 - m^2}$$

This can be varified by inserting the above in equation (9),

$$\begin{aligned}
-i\frac{\Big(g^{\mu\nu}(-p^2+m^2) + p^\mu p^\nu\Big)}{(p^2-m^2)}&\left(g_{\nu\lambda} - \frac{p_\nu p_\lambda}{m^2}\right)\\
&= +ig^\mu_\lambda - \frac{i}{p^2-m^2}\left[g^{\mu\nu}(-p^2+m^2)\left(\frac{-p_\nu p_\lambda}{m^2}\right) + p^\mu p^\nu g_{\nu\lambda} - \frac{p^\mu p^\nu p_\lambda}{m^2}\right]\\
&= +ig^\mu_\lambda - \frac{i}{p^2-m^2}\left[(-p^2+m^2)\left(\frac{-p_\nu p_\lambda}{m^2}\right)g^{\mu\nu} + p^\mu p^\nu g_{\nu\lambda} - \frac{p^2}{m^2}p^\mu p_\lambda\right]\\
&= +ig^\mu_\lambda - \frac{i}{p^2-m^2}\left[\frac{-p^2}{m^2}(-p^\mu p_\lambda) - p_\nu p_\lambda g^{\mu\nu} + p^\mu p^\nu g_{\nu\lambda} - \frac{p^2}{m^2}p^\mu p_\lambda\right]\\
&= +ig^\mu_\lambda - \frac{i}{p^2-m^2}\left[\frac{p^2}{m^2}p^\mu p_\lambda - \frac{p^2}{m^2}p^\mu p_\lambda - p_\nu p_\lambda g^{\mu\nu} + p^\mu p^\nu g_{\nu\lambda}\right]\\
&= +ig^\mu_\lambda - \frac{i}{p^2-m^2}\left[-p^\mu p_\lambda + p^\mu p_\lambda\right]\\
&= +ig^\mu_\lambda \quad \text{as we desired.}
\end{aligned}$$