Solution Tutorial 9: Modern Physics

1. Answer

For a $3D$ infinite well, we are given that

$$L = 1\,\text{cm}$$

According to Maxwell-Boltzmann distribution:

$$\frac{N_2}{N_1} = \exp\left(-\frac{\Delta E}{k_B T}\right)$$

If $\frac{N_2}{N_1} \approx 0$ i.e. all atoms in the ground state ($N_1 \to \infty$), then

$$\begin{aligned}
\Delta E &\gg k_B T \\
E_{211} - E_{111} &\gg k_B T \\
(2^2+1^2+1^2)\frac{\pi^2\hbar^2}{2mL^2} - (1^2+1^2+1^2)\frac{\pi^2\hbar^2}{2mL^2} &\gg k_B T \\
3\frac{\pi^2\hbar^2}{2mL^2} &\gg k_B T \\
T &\ll 3\frac{\pi^2\hbar^2}{2mL^2 k_B} \\
T &\ll \frac{3 \times (3.14)^2 \times (1.05 \times 10^{-34})^2}{2 \times 4 \times 1.6 \times 10^{-27} \times (0.01)^2 \times 1.38 \times 10^{-23}} \\
&\boxed{T \ll 1.846 \times 10^{-14}\,\text{K}}
\end{aligned}$$

2. Answer

It is given that

$$\begin{aligned}
V &= 10^{-15}\,\text{m}^{-3} \\
N &= 2000\,\text{atoms} \\
T &= ?
\end{aligned}$$

We know that

$$\begin{aligned}
r &< \lambda \\
r &< \frac{h}{mv} \qquad (2.1)
\end{aligned}$$

We know that

$$\begin{aligned}
\frac{1}{2}mv^2 &= \frac{3}{2}k_BT \\
v &= \sqrt{\frac{3k_BT}{m}} \\
mv &= m\sqrt{\frac{3k_BT}{m}} \\
mv &= \sqrt{3k_BTm}
\end{aligned}$$

Eq. (2.1) becomes now

$$\begin{aligned}
r &< \frac{h}{\sqrt{3k_BTm}} \\
r^2 &< \frac{h^2}{3k_BTm} \qquad (2.2)
\end{aligned}$$

We know that, volume can be represented in terms of number density $\left(n = \frac{N}{V}\right)$ as

$$\begin{aligned}
\frac{1}{n} &= \frac{4}{3}\pi r^3 \\
r &= \left(\frac{3}{4\pi n}\right)^{2/3}
\end{aligned}$$

Eq. (2.2) can be written as

$$\begin{aligned}
\left(\frac{3}{4\pi n}\right)^{2/3} &< \frac{h^2}{3k_BTm} \\
T &< \left(\frac{4\pi n}{3}\right)^{2/3} \frac{h^2}{3k_Bm} \\
T &< \left(\frac{4 \times 3.14 \times 2000}{3 \times 10^{-15}}\right)^{2/3} \frac{(6.626 \times 10^{-34})^2}{3 \times 1.38 \times 10^{-23} \times 87 \times 1.6 \times 10^{-27}} \\
&\boxed{T < 3.14 \times 10^{-7}\,\mathrm{K}}
\end{aligned}$$

3. Answer

Since

$$\begin{aligned}
r &> \lambda \\
r &> \frac{h}{mv} \qquad (3.1)
\end{aligned}$$

We know that

$$\begin{aligned}
\frac{1}{2}mv^2 &= \frac{3}{2}k_BT \\
v &= \sqrt{\frac{3k_BT}{m}} \\
mv &= m\sqrt{\frac{3k_BT}{m}} \\
mv &= \sqrt{3k_BTm}
\end{aligned}$$

Eq. (3.1) becomes now

$$\begin{aligned}
r &> \frac{h}{\sqrt{3k_BTm}} \\
r^2 &> \frac{h^2}{3k_BTm} \\
T &> \frac{h^2}{3k_Bmr^2} \\
T &> \frac{(6.626 \times 10^{-34})^2}{3 \times 1.38 \times 10^{-23} \times 9.1 \times 10^{-31} \times (0.3 \times 10^{-9})^2} \\
&\boxed{T > 1.29 \times 10^5\,\mathrm{K}}
\end{aligned}$$