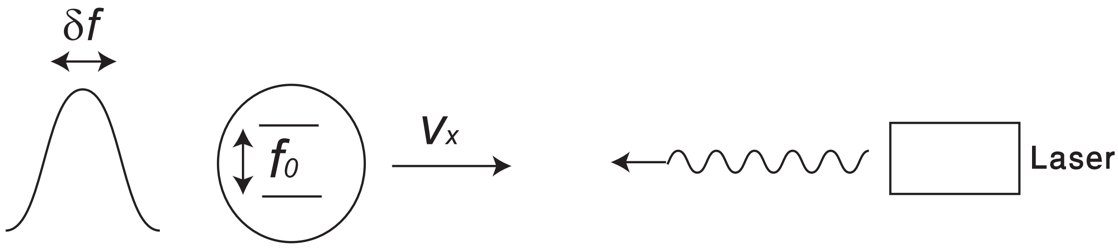
Solution Assignment 6: Modern Physics

1. This question deals with the Doppler cooling of sodium atoms. I am happy that you are beginning to understand the principles underlying monumental pieces of work!

   (a) A laser beam of frequency f_L propagates towards an atom moving with an initial speed v_x towards the laser source.

   The atomic level spacing is hf_0 (in energy) and $f_L = f_0 + \Delta f$, where Δf is a detuning factor. Using the Doppler mechanism discussed in class, what should be Δf for the atom to efficiently absorb the photon? [3 Marks]

   (b) What is the change in momentum of the atom (sign and magnitude) when the incoming photon is absorbed? [2 Marks]

   (c) What is the force and the corresponding acceleration (with sign)? The mass of the atom is m. The force acts for a time τ which is the lifetime of the excited state. [4 Marks]

   (d) After a time τ, (on average) the excited atom will emit a photon by spontaneous emission and will be available to absorb another photon from the incoming laser beam. Discuss why we need to change the detuning factor Δf for the second absorption and all subsequent absorptions? [2 Marks]

   (e) Estimate how much time (t_{min})is required to almost completely halt the atom? [3 Marks]

   (f) How far does the atom move in the time t_{min} calculated above? [3 Marks]

   (g) Discuss when will the cooling stop? The linewidth δf is naturally broadened by the uncertainty principle. [3 Marks]

   Hint: δf sets the limit on the minimum detuning factor Δf.

Answer 1

(a) The laser frequency in the lab frame is

$$f_\ell = f_0 + \Delta f,$$

The slightly detuned (by Δf) from the atomic transition f_0 apparent frequency seen by the atom is,

$$f' = \frac{f_\ell}{\left(1 - \dfrac{v_x}{c}\right)} = f_\ell\left(1 - \frac{v_x}{c}\right)^{-1}.$$

Using Binomial expension and assuming that $v_x \ll c$

$$f' \approx f_\ell\left(1 + \frac{v_x}{c}\right)$$

and upon substituting the value of f_ℓ into the above yields the frequency seen by the moving atom,

$$f' = (f_0 + \Delta f)\left(1 + \frac{v_x}{c}\right)$$

Since $\Delta f \ll 0$ and $\dfrac{v_x}{c} \ll 0$, we can ignore the term $\Delta f v_x/c$ yielding,

$$f' = f_0 + f_0\frac{v_x}{c} + \Delta f\ .$$

If $\Delta f = -f_0\dfrac{v_x}{c}$, then

$$f' = f_0$$

and efficient absorption will take place under this condition.

(b) The photon has a momentum $-h/\lambda$. When the photon is absorbed during an inelastic collision, $-h/\lambda$ disappears. This gives a kick to the atom in the negative direction, so

$$\Delta p_x = -\frac{h}{\lambda}\ .$$

(c) The force is given by

$$F_x = \frac{dp_x}{dt} \approx \frac{\Delta p_x}{\Delta t} \, .$$

We are provided that the force acts for a time τ, which is the lifetime of the excited state, so approximately

$$\begin{aligned} F &= \frac{-h}{\lambda \tau} \\ \text{As} \qquad a &= \frac{F}{m} \\ &= \frac{-h}{m\lambda\tau} \end{aligned}$$

Note that it is an approximate order of magnitude estimate. The negative sign shows deceleration.

(d) We derived an expression for detuning frequency in part (a) of this problem.

$$\Delta f = -f_0 \frac{v_x}{c} \, .$$

As it can be seen that the detuning frequency depends upon the speed of the atom as well. As atom is decelerating, the v_x undergoes a change after each absorption/emission cycle. So $(v_x)_{\text{new}}$ reduces from the original v_x after each absorption or emission cycle. The detuning frequency can be written as

$$\Delta f^n = -f_0 \frac{v_x^{(n)}}{c}$$

where $v_x^{(n)}$ is the velocity of the atom after n^{th} absorption/emission cycle.

(e) If the atom came to a complete rest, it means that its speed becomes zero. Atom was moving with some initial velocity v_x. We can use the equation of motion to compute the minimum time required:

$$\begin{aligned} v_f &= v_i + at \\ 0 &= v_x + \left(\frac{-h}{m\lambda\tau}\right) t_{\min} \\ t_{\min} &= \frac{v_x m \lambda \tau}{h} \, . \end{aligned}$$

(f) We could make use of the relevant equation of motion.

$$\begin{aligned} s &= v_i t + \frac{1}{2}at^2 \\ d_{\min} &= v_x\left(\frac{v_x\lambda m\tau}{h}\right) + \frac{1}{2}\left(-\frac{h}{m\lambda\tau}\right)\left(\frac{v_x\lambda m\tau}{h}\right)^2 \\ &= \left(\frac{v_x^2\lambda m\tau}{h}\right) - \left(\frac{h}{2m\lambda\tau}\right)\left(\frac{v_x^2\lambda^2 m^2\tau^2}{h^2}\right) \\ &= \left(\frac{v_x\lambda m\tau}{h}\right) - \left(\frac{v_x^2\lambda m\tau}{2h}\right) \\ &= \left(\frac{v_x^2\lambda m\tau}{2h}\right). \end{aligned}$$

(g) As v_x goes down, detuning frequency Δf goes down. So when detuning frequency reaches a value closer to $\delta f/2$ then cooling will be stopped. Approximate $T_{\min}$ can be found as

$$\begin{aligned} k_B T_{\min} &\approx \frac{h\,\delta f}{2} \\ T_{\min} &\approx \frac{h\,\delta f}{2k_B} = \frac{h}{2k_B\tau} \quad \text{since} \quad \tau = \frac{1}{\delta f}. \end{aligned}$$

This is when Δf approaches the uncertainty in f_0 (i.e. δf) itself!

2. This question explores a similar problem this time using numerical values to give you orders of magnitude estimates.

A beam of cesium atoms travelling in the $+x$ direction is emitted from an oven. The temperature is 200°C. A laser beam of wavelength 852 nm propagate in the $-x$ direction and is used to cool the atoms. The excited level has a life-time of 32 ns. The relative atomic mass of cesium is 132.9.

(a) Initially what is the detuning of the laser relative to the transition that produces efficient laser cooling? [2 Marks]

(b) What is the average momentum change imparted to a cesium atom during an absorption-emission cycle? What is the maximum decelerating force that will be exerted on the atoms? [2 Marks]

(c) Estimate the number of absorption emission cycles required to cool the atoms to their minimum temperature. Estimate the time taken for the atoms to reach this temperature, and the distance they would travel during the cooling process. [2 Marks]

(d) Calculate the final temperature that the atoms reach after this experiment. [2 Marks]

Answer 2

(a) As we derived a relation for detuning in Q1(a)

$$\Delta f = -f_0 \, \frac{v_x}{c}$$

We are provided the wavelength of the transit. The corresponding frequency f_0 can be found

$$\begin{aligned} f_0 &= c\lambda_0 \\ \Rightarrow \qquad \Delta f &= -\lambda_0 v_x \ . \end{aligned}$$

Now v_x can be found using the relation

$$\begin{aligned} \frac{1}{2}k_B T &= \frac{1}{2}mv_x^2 \\ \Rightarrow \qquad v_x &= \sqrt{\frac{k_B T}{m}} \\ &= \sqrt{\frac{(1.38 \times 10^{-23})(573)}{(132.9 \times 1.6 \times 10^{-27})}} \\ &= 192.84 \text{ m/s} \ . \end{aligned}$$

(b) We can make use of the expression derived in previous question

$$\Delta p_x = -\frac{h}{\lambda} = \frac{6.63 \times 10^{-34}}{852 \times 10^{-9}} = -7.78 \times 10^{-28} \text{ kg m/s}$$

The force can be written as,

$$|F| = \frac{\Delta p_x}{\tau} = \frac{7.78^{-28}}{32 \times 10^{-9}} = 2.34 \times 10^{-20} \text{ N.}$$

(c) When atom cools down to its minimum temperature it would come to stop. So we can use the expression derived in Q1(e).

$$\begin{aligned}
t_{\min} &= \left(\frac{v_x m\lambda}{h}\right)\tau \\
&= n\tau \\
n &= \left(\frac{v_x m\lambda}{h}\right) \\
&= \frac{(192.84)(212.64\times 10^{-27})(852\times 10^{-9})}{(6.63\times 10^{-34})} \\
&\approx 529695
\end{aligned}$$

So we need almost half a million absorption/emission cycles to to cool the atoms to their minimum temperature.

Time taken to reach this temperature would be

$$t_{\min} \approx n\tau = 0.045 \text{ s} = 45 \text{ ms}.$$

Courtesy the shorter life-time, we achieved a minimum temperature in merely 45 ms. It is faster than the blink of an eye! We can make use of our relation derived before

$$\begin{aligned}
d_{\min} &= \frac{v_x^2 m\lambda\tau}{2h} \\
&= \frac{(192.84)^2(132.9\times 1.6\times 10^{-27})(852\times 10^{-9})(32\times 10^{-9})}{(2\times 6.63\times 10^{-34})} \\
&= 16 \text{ cm}.
\end{aligned}$$

(d)

$$T_{\min} = \frac{h}{2k_B\tau} = \frac{6.63\times 10^{-34}}{2\times 1.38\times 10^{-23}\times 32\times 10^{-9}} = 7.5\times 10^{-4} \text{ K}.$$

3. (a) Consider a gas of hydrogen at room temperature= 300 K. What fraction of atoms would have the first excited state populated with respect to the ground state? Assume equilibrium. [3 Marks]

   (b) At what temperature could I expect that 1% of the atoms would be in the first excited state and 99% in the ground state? [3 Marks]

Answer 3

(a)

$$\begin{aligned}
\frac{N_2}{N_1} &= \exp\left(\frac{-\Delta E}{k_B T}\right) \\
\Delta E &= E_2 - E_1 = \frac{E_1}{2^2} - E_1 = \frac{-3E_1}{4} \\
E_1 &= -13.6 \text{ eV} \\
\Delta E &= 10.2 \text{ eV} \\
\Rightarrow \qquad \frac{N_2}{N_1} &= \exp\left(\frac{-10.2 \times 1.6 \times 10^{-19}}{1.38 \times 10^{-23} \times 300}\right) \\
&= 6.3 \times 10^{-172}
\end{aligned}$$

This number tells us that there are almost no atoms in the excited state at room temperature.

(b) Now if we want to take the 1% population of hydrogen atoms in an excited state,

$$\begin{aligned}
\frac{N_2}{N_1} &= \exp\left(\frac{-\Delta E}{k_B T}\right) \\
\frac{-\Delta E}{k_B T} &= \ln\left(\frac{N_2}{N_1}\right) \\
T &= \frac{-\Delta E}{k_B \ln\left(\frac{N_2}{N_1}\right)} \\
&= \frac{-10.2 \times 1.6 \times 10^{-19}}{1.38 \times 10^{-23} \times -4.595} \\
&= 25736 \text{ K}
\end{aligned}$$

that is an incredible amount of temperature. The surface temperature of Sun is merely 6000 K. Temperature at the core of the sun is equal to 15 million K.

4. I have a gas inside a sealed chamber. The line spectrum contains lines from various transitions. As I heat the chamber, all lines get broader. What could be the possible mechanism for this? [5 Marks]

Answer 4

According to Charle's law, with a rise in temperature, pressure rises too. This leads to more collisions. As more and more atoms collied, they fall to lower energy states,

which results in decreased life time τ. Hence the lines get broader. This phenomenon is called pressure broadening.