Solution Assignment 11: Quantum Field Theory
Due Date: 7 May. 4 pm

1. Starting with

$$G^+(x,y) = \Theta(x-y)\, \langle x|\, e^{-i\hat{H}(t_x - t_y)}\, |y\rangle$$

compute the Green function for a free particle with end points $\vec{x} = \vec{y} = 0$. Show that this equals:

$$G^+(x=0, y=0) = \Theta(x-y)\left(\frac{-im}{2\pi T}\right)^{1/2}.$$

Note that $\hat{H} = \hat{p}^2/2m$ and $T = t_x - t_y$. [5 **Marks**]

Answer 1

$$G^+(x,y) = \Theta(x^0 - y^0)\, \langle x|\, e^{-i\frac{\hat{p}^2}{2m}(t_x - t_y)}\, |y\rangle$$

since

$$\begin{aligned}
|y\rangle &= \int dp\ \ |p\rangle\, \langle p|y\rangle \\
&= \int \frac{dp}{(2\pi)^{1/2}}\ e^{-ipy}\, |p\rangle
\end{aligned}$$

we have

$$\begin{aligned}
G^+(x,y) &= \Theta(x^0 - y^0) \int \frac{dp\ dq}{2\pi}\ e^{+i(qx - py)}\, \langle q|\, e^{-i\frac{\hat{p}^2}{2m}T}\, |p\rangle \\
&= \Theta(x^0 - y^0) \int \frac{dp\ dq}{2\pi}\, \langle q|p\rangle\ \ e^{-i\frac{\hat{p}^2}{2m}T}\ e^{+ip(x-y)}
\end{aligned}$$

Since $\langle q|p\rangle = \delta(q-p)$

$$\begin{aligned}
G^+(x,y) &= \Theta(x^0 - y^0) \int \frac{dp}{2\pi}\ e^{-i\frac{\hat{p}^2}{2m}T}\ e^{+ip(x-y)} \\
G^+(x=0, y=0) &= \Theta(x^0 - y^0) \int \frac{dp}{2\pi}\ e^{-i\frac{\hat{p}^2}{2m}T}
\end{aligned}$$

Using

$$\int dx\ e^{-\frac{ax^2}{2}} = \sqrt{\frac{2\pi}{a}}$$

we have

$$G^+(0,0) = \Theta(x^0 - y^0)\left(\frac{-i2\pi m}{T}\right)^{1/2} \frac{1}{2\pi}$$
$$G^+(0,0) = \Theta(x^0 - y^0)\left(\frac{-im}{2\pi T}\right)^{1/2} \quad \text{as desired.}$$

2. Using

$$G(x,y) = T\langle 0|\, \psi(\hat{x})\hat{\bar{\psi}}(y)\,|0\rangle$$

and

$$\hat{\psi}(x) = \int \frac{d^3p}{(2\pi)^{3/2}(2E_{\mathbf{p}})^{1/2}} \sum_s \left(u^s(p)\hat{a}_{s\mathbf{p}} e^{-ip\cdot x} + v^s(p)\hat{b}^\dagger_{s\mathbf{p}} e^{+ip\cdot x}\right)$$

show a complete, neat and systmetic derivation of the fermionic free propagator

$$G_0(x,y) = \int \frac{d^4p}{(2\pi)^4}\; i\; \frac{\not{p} + m}{p^2 - m^2 + i0^+}\; e^{-ip\cdot(x-y)}$$

[15 **Marks**]

Answer 2

$$T\langle 0|\, \hat{\psi}(x)\hat{\bar{\psi}}(y)\,|0\rangle = \underbrace{\Theta(x^0 - y^0)\,\langle 0|\, \hat{\psi}(x)\hat{\bar{\psi}}(y)\,|\,0\rangle}_{I} - \underbrace{\Theta(y^0 - x^0)\,\langle 0|\, \hat{\bar{\psi}}(y)\hat{\psi}(x)\,|\,0\rangle}_{II}$$

Now

$$\hat{\psi}(x) = \int \frac{d^3p}{(2\pi)^{3/2}(2E_{\mathbf{p}})^{1/2}} \sum_s \left(u^s(p)\hat{a}_{s\mathbf{p}} e^{-ip\cdot x} + v^s(p)\hat{b}^\dagger_{s\mathbf{p}} e^{+ip\cdot x}\right)$$

I:

$$\langle 0|\, \hat{\psi}(x)\hat{\bar{\psi}}(y)\,|0\rangle = \langle 0| \int \frac{d^3p\; d^3q}{(2\pi)^3(2E_{\mathbf{p}})^{1/2}(2E_{\mathbf{q}})^{1/2}} \sum_{r,s} \left(u^s(p)\hat{a}_{s\mathbf{p}} e^{-ip\cdot x} + v^s(p)\hat{b}^\dagger_{s\mathbf{p}} e^{+ip\cdot x}\right)$$
$$\left(\bar{u}^r(q)\hat{a}^\dagger_{r\mathbf{q}} e^{+iq\cdot y} + \bar{v}^r(q)\hat{b}_{r\mathbf{q}} e^{-iq\cdot y}\right) |0\rangle$$

we have $\langle 0|\, \hat{a}_{s\mathbf{p}}\hat{a}^\dagger_{r\mathbf{q}}\,|0\rangle = S^{(3)}(\mathbf{p}-\mathbf{q})\delta_{r,s}$

$$\Rightarrow \qquad \langle 0|\, \hat{\psi}(x)\hat{\bar{\psi}}(y) = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3 E_{\mathbf{p}}} \sum_s u^s(p)\bar{u^s}(p) e^{-ip\cdot(x-y)}$$

II:

$$
\begin{aligned}
\langle 0|\,\hat{\bar{\psi}}(y)\hat{\psi}(x)\,|0\rangle &= \langle 0|\,\frac{1}{2}\int \frac{d^3p\ d^3q}{(2\pi)^3(2E_{\mathbf{p}})^{1/2}(2E_{\mathbf{q}})^{1/2}}\left(\bar{u}^s(p)\hat{a}^\dagger_{s\mathbf{p}}e^{+ip\cdot y}+\bar{v}^s(p)\hat{b}_{s\mathbf{p}}e^{-ip\cdot y}\right)\\
&\quad\left(\bar{u}^r(q)\hat{a}_{r\mathbf{q}}e^{-iq\cdot x}+\bar{v}^r(q)\hat{b}^\dagger_{r\mathbf{q}}e^{+iq\cdot x}\right)|0\rangle\\
&= \frac{1}{2}\int \frac{d^3p}{(2\pi)^3E_{\mathbf{p}}}\left(\bar{v}^s(p)v^s(p)e^{+ip\cdot(x-y)}\right)\\
T\,\langle 0|\,\hat{\psi}(x)\hat{\bar{\psi}}(y)\,|0\rangle &= \Theta(x^0-y^0)\frac{1}{2}\int \frac{d^3p}{(2\pi)^3E_{\mathbf{p}}}\sum_s u^s(p)\bar{u}^s(p)e^{-ip\cdot(x-y)}\\
&- \Theta(y^0-x^0)\frac{1}{2}\int \frac{d^3p}{(2\pi)^3E_{\mathbf{p}}}\sum_s \bar{v}^s(p)v^s(p)e^{+ip\cdot(x-y)}\\
&= \Theta(x^0-y^0)\frac{1}{2}\int \frac{d^3p}{(2\pi)^3E_{\mathbf{p}}}(\gamma\cdot p+m)e^{-ip\cdot(x-y)}\\
&- \Theta(y^0-x^0)\frac{1}{2}\int \frac{d^3p}{(2\pi)^3E_{\mathbf{p}}}(\gamma\cdot p-m)e^{+ip\cdot(x-y)}\\
\gamma\cdot p &= \gamma^0p_0+\gamma^1p_1+\gamma^2p_2+\gamma^3p_3 = \not{p}\\
\not{p} &= \gamma^\mu p_\mu\\
\Rightarrow\qquad T\,\langle 0|\,\hat{\psi}(x)\hat{\bar{\psi}}(y)\,|0\rangle &= \Theta(x^0-y^0)\frac{d^3p}{(2\pi)^3}\ \frac{\not{p}+m}{2E_{\mathbf{p}}}\ e^{-ip\cdot(x-y)}\\
&- \Theta(y^0-x^0)\frac{d^3p}{(2\pi)^3}\ \frac{\not{p}-m}{2E_{\mathbf{p}}}\ e^{+ip\cdot(x-y)}\\
\Theta(t) &= \frac{i}{2\pi}\int_{z=-\infty}^{\infty}\frac{e^{-izt}}{z+i0^+}\ dz\\
T\,\langle 0|\,\hat{\psi}(x)\hat{\bar{\psi}}(y)\,|0\rangle &= \frac{i}{2\pi}\int_{z=-\infty}^{\infty}\frac{dz\ d^3p}{(2\pi)^3}\ \frac{\not{p}+m}{2E_{\mathbf{p}}(z+i0^+)}\ e^{-iz\cdot(x^0-y^0)}\ e^{-ip\cdot(x-y)}\\
&- \frac{i}{2\pi}\int_{z=-\infty}^{\infty}\frac{dz\ d^3p}{(2\pi)^3}\ \frac{\not{p}-m}{2E_{\mathbf{p}}(z+i0^+)}\ e^{+iz\cdot(x^0-y^0)}\ e^{+ip\cdot(x-y)}\\
&= \frac{i}{2\pi}\int_{z=-\infty}^{\infty}\frac{dz\ d^3p}{(2\pi)^3}\ \frac{\not{p}+m}{2E_{\mathbf{p}}(z+i0^+)}\ e^{-i(E_{\mathbf{p}}+z)\cdot(x^0-y^0)}\ e^{-ip\cdot(x-y)}\\
&- \frac{i}{2\pi}\int_{z=-\infty}^{\infty}\frac{dz\ d^3p}{(2\pi)^3}\ \frac{\not{p}-m}{2E_{\mathbf{p}}(z+i0^+)}\ e^{i(E_{\mathbf{p}}+z)\cdot(x^0-y^0)}\ e^{-ip\cdot(x-y)}\\
&= i\int_{-\infty}^{\infty}\frac{d^4p}{(2\pi)^4}\ \frac{\not{p}+m}{2E_{\mathbf{p}}(p^0-E_{\mathbf{p}}+i0^+)}\ e^{-ip\cdot(x-y)}\\
&- i\int_{-\infty}^{\infty}\frac{d^4p}{(2\pi)^4}\ \frac{\not{p}-m}{2E_{\mathbf{p}}(p^0-E_{\mathbf{p}}+i0^+))}\ e^{+ip\cdot(x-y)}
\end{aligned}
$$

In the second term $p \mapsto -p$, $p^0 \mapsto -p^0$

$$
\begin{aligned}
\Delta_F(x-y) &= i\int_{-\infty}^{\infty} \frac{d^4p}{(2\pi)^4}\ \frac{\not{p}+m}{2E_{\mathbf{p}}(p^0 - E_{\mathbf{p}} + i0^+))}\ e^{+ip\cdot(x-y)} \\
&- i\int_{-\infty}^{\infty} \frac{d^4p}{(2\pi)^4}\ \frac{-\not{p}-m}{2E_{\mathbf{p}}(-p^0 - E_{\mathbf{p}} + i0^+))}\ e^{-ip\cdot(x-y)} \\
&= \int_{-\infty}^{\infty} \frac{d^4p}{(2\pi)^4}\ \frac{i}{2E_{\mathbf{p}}} e^{-ip\cdot(x-y)} \left[\frac{\not{p}+m}{p^0 - E_{\mathbf{p}} + i0^+} + \frac{-\not{p}-m}{p^0 + E_{\mathbf{p}} + i0^+}\right] \\
&= \int_{-\infty}^{\infty} \frac{d^4p}{(2\pi)^4}\ \frac{i}{2E_{\mathbf{p}}} e^{-ip\cdot(x-y)} (\not{p}+m) \left[\frac{1}{p^0 - E_{\mathbf{p}} + i0^+} - \frac{1}{p^0 + E_{\mathbf{p}} + i0^+}\right] \\
&= \int_{-\infty}^{\infty} \frac{d^4p}{(2\pi)^4}\ \frac{i}{2E_{\mathbf{p}}} e^{-ip\cdot(x-y)} (\not{p}+m) \left[\frac{p^0 + E_{\mathbf{p}} + i0^+ - p^0 + E_{\mathbf{p}} - i0^+}{(p^0)^2 - (E_{\mathbf{p}})^2 + i0^+}\right] \\
&= \int_{-\infty}^{\infty} \frac{d^4p}{(2\pi)^4}\ \frac{i(\not{p}+m)}{(p^0)^2 - (E_{\mathbf{p}})^2 + i0^+} e^{-ip\cdot(x-y)} \\
(p^0)^2 &= p^2 + |\mathbf{p}|^2 \\
p^2 &= (p^0)^2 - |\mathbf{p}|^2 \\
\text{Now} \quad E_{\mathbf{p}}^2 &= m^2 + |\mathbf{p}|^2 \\
|\mathbf{p}|^2 &= E_{\mathbf{p}}^2 - m^2 \\
p^2 - E_{\mathbf{p}}^2 &= p^2 + |\mathbf{p}|^2 - m^2 - |\mathbf{p}|^2 \\
&= p^2 - m^2. \\
\Delta_F(x,y) = G(x-y) &= \int_{-\infty}^{\infty} \frac{d^4p}{(2\pi)^4}\ \frac{i(\not{p}+m)}{p^2 - m^2 + i0^+} e^{-ip\cdot(x-y)}
\end{aligned}
$$

3. In the minimal coupling technique, ordinary derivatives are replaced by covariant derivatives. Apply this to a complex massive scalar field and a non-interacting electromagnetic field. The resulting theory is not only gauge invariant but leads to an interaction between the complex scalar field and the electromagnetic field. What is the interaction term? [5 **Marks**]

Answer 3

$$
\mathscr{L} = (\partial^\mu \psi)^\dagger (\partial_\mu \psi) - m^2 \psi^\dagger \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}
$$

Applying the minimal coupling prescription, (only to the massive field)

$$
\mathscr{L} = (D^\mu \psi)^\dagger (D_\mu \psi) - m^2 \psi^\dagger \psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}
$$

Since $D_\mu = \partial_\mu + iqA_\mu$, we have

$$\begin{aligned}
(D^\mu\psi)^\dagger &= \Big((\partial^\mu + iqA^\mu)\psi\Big)^\dagger \\
&= \partial^\mu\psi^\dagger - iqA^\mu\psi^\dagger \\
(D_\mu\psi) &= (\partial_\mu + iqA_\mu)\psi \\
&= \partial_\mu\psi + iqA_\mu\psi
\end{aligned}$$

So

$$\begin{aligned}
\mathscr{L} &= (\partial^\mu\psi^\dagger - iqA^\mu\psi^\dagger)(\partial_\mu\psi + iqA_\mu\psi) - m^2\psi^\dagger\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\
&= (\partial^\mu\psi^\dagger)(\partial_\mu\psi) - m^2\psi^\dagger\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + iq(\partial^\mu\psi^\dagger)(A_\mu\psi) - iq(A^\mu\psi^\dagger)(\partial_\mu\psi) + q^2A^\mu\psi^\dagger A_\mu\psi
\end{aligned}$$

The last three terms represent the coupling of the scalar field with the electro magnetic field.

4. (a) Show that the momentum space representation of the fermionic free propagation

$$\widetilde{G}(p) = i\ \frac{\not{p} + m}{p^2 - m^2 + i0^+}$$

can be written as

$$\widetilde{G}(p) = \frac{i}{\not{p} - m + i0^+}\ .$$

[5 **Marks**]

(b) The free propagator

$$G = \int \frac{d^4p}{(2\pi)^4}\ \frac{i}{\not{p} - m + i0^+}\ e^{-ip\cdot(x-y)}$$

is a Green's function of which operator? Substantiate your answer by a calculation. [5 **Marks**]

Answer 4

(a)

$$\widetilde{G}p = i\ \frac{\not{p} + m}{p^2 - m^2 + i0^+} \tag{1}$$

$$\begin{aligned}
(\not{p} + m)(\not{p} - m) &= \not{p}^2 - \not{p}m + \not{p}m - m^2 \\
&= \not{p}^2 - m^2 \\
&= \gamma^\mu p_\mu \gamma^\nu p_\nu - m^2 \\
&= \gamma^\mu \gamma^\nu p_\mu p_\nu - m^2
\end{aligned}$$

$$\text{Now} \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$
$$\Rightarrow \quad \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$
$$\gamma^\mu\gamma^\nu p_\mu p_\nu = \gamma^\nu\gamma^\mu p_\nu p_\mu = \gamma^\nu\gamma^\mu p_\mu p_\nu$$

Since $p_\mu p_\nu$ is symmetric with respect to swapping of μ and ν. So we can write $\gamma^\nu\gamma^\mu p_\mu p_\nu$ as

$$\frac{1}{2}\{\gamma^\mu, \gamma^\nu\}p_\mu p_\nu = g^{\mu\nu}p_\mu p_\nu$$
$$\therefore \quad (\not{p} + m)(\not{p} - m) = g^{\mu\nu}p_\mu p_\nu - m^2 = p^2 - m^2.$$

Inserting this into the denomenator of equation (1) yields

$$\widetilde{G}(p) = \frac{i}{\not{p} - m + i0^+} \cdot$$

(b)

$$(i\not{\partial} - m)\int \frac{d^4p}{(2\pi)^4}\ \frac{i}{\not{p} - m + i0^+}\ e^{-ip\cdot(x-y)}$$

choose to calculate

$$\begin{aligned}\not{\partial}\int \frac{d^4p}{(2\pi)^4}\ \frac{i}{\not{p} - m}\ e^{-ip\cdot(x-y)} &= \int \frac{d^4p}{(2\pi)^4}\ \frac{i}{\not{p} - m}\ \not{\partial}\ e^{-ip\cdot(x-y)} \\ &= \int \frac{d^4p}{(2\pi)^4}\ \frac{i}{\not{p} - m}\ \gamma^\mu\ \partial_\mu\ e^{-ip\cdot(x-y)}\end{aligned}$$

Now

$$\begin{aligned}\partial_\mu\ e^{-ip\cdot(x-y)} &= e^{-ip\cdot(x-y)}\ (-ip_\mu) \\ \not{\partial}\int \frac{d^4p}{(2\pi)^4}\ \frac{i}{\not{p} - m}\ e^{-ip\cdot(x-y)} &= -i\int \frac{d^4p}{(2\pi)^4}\ \frac{i\not{p}}{\not{p} - m}\ e^{-ip\cdot(x-y)} \\ (i\not{\partial} - m)\int \frac{d^4p}{(2\pi)^4}\ \frac{i}{\not{p} - m}\ e^{-ip\cdot(x-y)} &= \int \frac{d^4p}{(2\pi)^4}\ \frac{i(\not{p} - m)}{\not{p} - m}\ e^{-ip\cdot(x-y)} \\ &= i\int \frac{d^4p}{(2\pi)^4}\ e^{-ip\cdot(x-y)} \\ &= i\delta^{(4)}(x - y).\end{aligned}$$

5. (a) Find the conserved Noether current for the massless Dirac equation. [5 **Marks**]

(b) Consider the Global chiral symmetry

$$\psi \mapsto e^{-i\alpha\gamma^5}\psi$$

where γ^5 is the chirality operator. Show that the massless Dirac Lagrangian is invariant under the chiral transformation. [5 **Marks**]

(c) What is the corresponding Noether current? [5 **Marks**]

Answer 5

(a)

$$\mathscr{L}_D = \bar{\psi}(i\partial\!\!\!/)\psi = \bar{\psi}(i\gamma^\mu\partial_\mu\psi)$$

The system has $U(1)$ global symmetry.

$$\begin{aligned}
\Pi^\mu_\psi &= \frac{\partial\mathscr{L}_D}{\partial(\partial_\mu\psi)} = i\bar{\psi}\gamma^\mu\partial_\mu = i\bar{\psi}\partial\!\!\!/ \\
\Pi^\mu_{\bar{\psi}} &= 0 \\
D\psi &= i\psi \\
J^\mu_N &= \Pi^\mu_\psi D\psi = i\bar{\psi}\partial\!\!\!/(i\psi) = -\bar{\psi}\partial\!\!\!/\psi = -\bar{\psi}\gamma^\mu\partial_\mu\psi
\end{aligned}$$

(b)

$$\begin{aligned}
\mathscr{L}_D &= \bar{\psi}(i\partial\!\!\!/)\psi = \bar{\psi}(i\gamma^\mu\partial_\mu\psi) \\
\psi(x) &\mapsto e^{-i\alpha\gamma^5}\psi(x) \\
\text{where} \qquad \gamma^5 &= i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}.
\end{aligned}$$

Under the chiralioty transformation,

$$\begin{aligned}
\psi^\dagger(x) &\mapsto \psi^\dagger(x)e^{-i\alpha\gamma^5} \\
\bar{\psi}(x) &\mapsto \psi^\dagger(x)e^{-i\alpha\gamma^5}\gamma^0 \\
\mathscr{L}'_D &= \psi^\dagger(x)e^{-i\alpha\gamma^5}\gamma^0(i\gamma^\mu\partial_\mu)(e^{i\alpha\gamma^5}\psi) \\
&= i\psi^\dagger(x)e^{-i\alpha\gamma^5}\gamma^0\ \gamma^\mu e^{i\alpha\gamma^5}\ \partial_\mu\psi
\end{aligned}$$

We know that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

We have

$$\begin{aligned}\{\gamma^5, \gamma^0\} &= i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3 + i\gamma^0\gamma^0\gamma^1\gamma^2\gamma^3\\ &= i\gamma^1\gamma^2\gamma^3 + i\gamma^1\gamma^2\gamma^3 = i\gamma^1(\gamma^2\gamma^3 + \gamma^2\gamma^3)\\ &= i\gamma^1(\gamma^2\gamma^3 + \gamma^3\gamma^2) = i\gamma^1\{\gamma^2, \gamma^3\} = 0\end{aligned}$$

So we can swap the order of γ^5 and γ^0 terms. Likewise for $i = 1, 2, 3$

$$\{\gamma^5, \gamma^i\} = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^i + i\gamma^i\gamma^0\gamma^1\gamma^2\gamma^3$$

Since $\{\gamma^i, \gamma^i\} = 2 \quad \Rightarrow \quad (\gamma^i)^2 = 1$, the anti commutator above will have two γ^i's removed from the terms as they multiply togather to yield identity. Suppose $i = 3$, the above anticommutator becomes

$$i\gamma^0\gamma^1\gamma^2 + i\gamma^0\gamma^1\gamma^3 = i\gamma^0\{\gamma^1, \gamma^2\} = 0$$

γ^5 and γ^3 can be swapped. Similarly γ^5 and γ^2 and γ^5 and γ^1 can be mutually swapped. So γ^5 swaps will all γ^μ's. This allows us to write the lagrangian density as:

$$\mathscr{L}'_D = i\psi^\dagger\gamma^0\gamma^\mu e^{-i\alpha\gamma^5}\partial_\mu\psi = i\bar{\psi}\gamma^\mu\partial_\mu\psi = -\bar{\psi}\partial\!\!\!/\psi = \mathscr{L}.$$

(c)

$$\begin{aligned}\mathscr{L}'_D &= i\bar{\psi}\partial\!\!\!/\psi = i\bar{\psi}\gamma^\mu\partial_\mu\psi\\ \Pi^\mu_\psi &= i\bar{\psi}\gamma^\mu, \qquad \Pi^\mu_{\bar{\psi}} = 0\end{aligned}$$

Under the chirality transformation,

$$\psi \mapsto e^{-i\alpha\gamma^5}\psi \approx (1 - i\alpha\gamma^5)\psi$$

So

$$\begin{aligned}D\psi &= -i\gamma^5\psi\\ J^\mu_N &= \Pi^\mu_\psi D\psi = i\bar{\psi}\gamma^\mu(-i\gamma^5\psi) = \bar{\psi}\gamma^\mu\gamma^5\psi\ .\end{aligned}$$