Solution Assignment 10: Quantum Field Theory
Due Date: 2 May. 10 am

1. When a fermion is at rest, the Dirac spinor for the particle is

$$u(p^0) = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}.$$

The left and right handed components are equal.

(a) Show that $(\gamma^0 - 1)u(p^0) = 0$. **[3 Marks]**

(b) Show that $e^{+i\mathbf{K}\cdot\phi}\gamma^0 e^{-i\mathbf{K}\cdot\phi} = \dfrac{p}{m}$. **[7 Marks]**

Answer 1

(a)

$$\gamma^0 - 1 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} - \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}$$
$$= \begin{pmatrix} -\mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{pmatrix}$$
$$(\gamma^0 - 1)u(p) = \sqrt{m}\begin{pmatrix} -\mathbf{1} & \mathbf{1} \\ \mathbf{1} & -\mathbf{1} \end{pmatrix}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$$
$$= \sqrt{m}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$$

Therefore $(\gamma^0 - 1)u(p^0) = 0$ when the fermion is at rest.

(b)

$$e^{i\mathbf{K}\cdot\phi} = \begin{pmatrix} e^{\frac{-\sigma\cdot\phi}{2}} & 0 \\ 0 & e^{\frac{+\sigma\cdot\phi}{2}} \end{pmatrix}$$
$$e^{+i\mathbf{K}\cdot\phi}\,\gamma^0\,e^{+i\mathbf{K}\cdot\phi} = \begin{pmatrix} e^{\frac{-\sigma\cdot\phi}{2}} & 0 \\ 0 & e^{\frac{+\sigma\cdot\phi}{2}} \end{pmatrix}\begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}\begin{pmatrix} e^{\frac{+\sigma\cdot\phi}{2}} & 0 \\ 0 & e^{\frac{-\sigma\cdot\phi}{2}} \end{pmatrix}$$

$$
\begin{aligned}
&= \begin{pmatrix} e^{\frac{-\sigma\cdot\phi}{2}} & 0 \\ 0 & e^{\frac{+\sigma\cdot\phi}{2}} \end{pmatrix} \begin{pmatrix} 0 & e^{\frac{-\sigma\cdot\phi}{2}} \\ e^{\frac{+\sigma\cdot\phi}{2}} & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & e^{-\sigma\cdot\phi} \\ e^{+\sigma\cdot\phi} & 0 \end{pmatrix}
\end{aligned}
$$

Now

$$
e^{+\sigma\cdot\phi} = \mathbf{1} + \sigma\cdot\phi + \frac{1}{2!}(\sigma\cdot\phi)^2 + \frac{1}{3!}(\sigma\cdot\phi)^3 + \frac{1}{4!}(\sigma\cdot\phi)^4 + \frac{1}{5!}(\sigma\cdot\phi)^5
$$

Let's work at these terms one by one:

$$
\begin{aligned}
\sigma\cdot\phi &= \sigma\cdot\frac{\phi}{|\phi|}\phi \\
&= \sigma\cdot\hat{\phi}\phi \\
&= \sigma_\phi\phi
\end{aligned}
$$

where σ_ϕ is the projection of the pauli vector along ϕ. Note $(\sigma_\phi)^2 = (\sigma_\phi)^4 = \mathbf{1}$ and so on. Therefore

$$
\begin{aligned}
e^{\sigma\cdot\phi} &= 1 + \sigma_\phi\phi + \frac{1}{2!}\phi^2 + \frac{1}{3!}\sigma_\phi\phi^3 + \ldots \\
&= \left(1 + \frac{1}{2!}\phi^2 + \frac{1}{4!}\phi^4 + \ldots\right) + \sigma_\phi\left(\phi + \frac{1}{3!}\phi^3 + \frac{1}{5!}\phi^5 + \ldots\right) \\
&= \cosh(\phi)1 + \sigma_\phi\sinh(\phi) \\
&= \cosh(\phi)1 + \sigma\cdot\frac{\phi}{|\phi|}\sinh(\phi)
\end{aligned}
$$

Now

$$
\begin{aligned}
\cosh(\phi) &= \frac{E_{\mathbf{p}}}{m}, \\
\sinh(\phi) &= \frac{|\mathbf{p}|}{m} \\
\therefore\ e^{\sigma\cdot\phi} &= \frac{E_{\mathbf{p}}}{m} + \frac{\sigma\cdot\phi}{m}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
e^{-\sigma\cdot\phi} &= \frac{E_{\mathbf{p}}}{m} - \frac{\sigma\cdot\phi}{m}. \\
e^{+i\mathbf{K}\cdot\phi}\,\gamma^0\,e^{+i\mathbf{K}\cdot\phi} &= \begin{pmatrix} 0 & \frac{E_{\mathbf{p}}-\sigma\cdot\phi}{m} \\ \frac{E_{\mathbf{p}}+\sigma\cdot\phi}{m} & 0 \end{pmatrix}.
\end{aligned}
$$

Now

$$\begin{aligned}
\not{p} &= \gamma^\mu p_\mu \\
&= \gamma^0 p_0 + \gamma^1 p_1 + \gamma^2 p_2 + \gamma^3 p_3 \\
&= \begin{pmatrix} 0 & p^0 - \sigma\cdot\mathbf{p} \\ p^0 + \sigma\cdot\mathbf{p} & 0 \end{pmatrix} \\
\therefore\ e^{+i\mathbf{K}\cdot\phi}\gamma^0 e^{-i\mathbf{K}\cdot\phi} &= \frac{\not{p}}{m}
\end{aligned}$$

2. (a) Prove that $u^\dagger(p)u(p) = 2E_{\mathbf{p}}\xi^\dagger\xi$. where

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi \\ \sqrt{p\cdot\bar{\sigma}}\ \xi \end{pmatrix}$$

is the momentum space solution of the Dirac equation. [5 **Marks**]

(b) Show that $\bar{u}(p)u(p) = 2m\xi^\dagger\xi$, where $\bar{u}(p) = u^\dagger(p)\gamma^0$. [5 **Marks**]

Answer 2

(a)

$$u^\dagger(p)\ u(p) = (\sqrt{p\cdot\sigma}\ \xi^\dagger\ \ \sqrt{p\cdot\bar{\sigma}}\ \xi^\dagger) \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi \\ \sqrt{p\cdot\bar{\sigma}}\ \xi^\dagger \end{pmatrix}$$

Now

$$\begin{aligned}
p\cdot\sigma &= p^0\sigma^0 - \mathbf{p}\cdot\sigma \\
p\cdot\bar{\sigma} &= p^0\sigma^0 + \mathbf{p}\cdot\sigma \\
p\cdot\sigma\ \xi^\dagger\xi + p\cdot\bar{\sigma}\ \xi^\dagger\xi &= (p\cdot\sigma + p\cdot\bar{\sigma})\xi^\dagger\xi \\
&= (p^0\sigma^0 - \mathbf{p}\cdot\sigma + p^0\sigma^0 + \mathbf{p}\cdot\sigma)\xi^\dagger\xi \\
&= 2p_0\xi^\dagger\xi \\
&= 2E_{\mathbf{p}}\xi^\dagger\xi.
\end{aligned}$$

(b)

$$\begin{aligned}
\bar{u}(p)\ u(p) &= u^\dagger(p)\ \gamma^0\ u(p) \\
&= (\sqrt{p\cdot\sigma}\ \xi^\dagger\ \ \sqrt{p\cdot\bar{\sigma}}\ \xi^\dagger) \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{p\cdot\sigma}\ \xi \\ \sqrt{p\cdot\bar{\sigma}}\ \xi^\dagger \end{pmatrix}
\end{aligned}$$

$$
\begin{aligned}
&= (\sqrt{p\cdot\sigma}\;\xi^\dagger\;\;\sqrt{p\cdot\bar{\sigma}}\;\xi^\dagger) \begin{pmatrix} \sqrt{p\cdot\bar{\sigma}}\;\xi \\ \sqrt{p\cdot\sigma}\;\xi^\dagger \end{pmatrix} \\
&= \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}}\xi^\dagger\xi + \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}}\xi^\dagger\xi \\
&= 2\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}}\xi^\dagger\xi \\
&= 2\sqrt{(p^0\sigma^0 - \mathbf{p}\cdot\sigma)(p^0\sigma^0 + \mathbf{p}\cdot\sigma)}\xi^\dagger\xi \\
&= 2\sqrt{E_{\mathbf{p}}^2 - |\mathbf{p}|^2}\;\xi^\dagger\xi \\
&= 2m\;\xi^\dagger\xi \\
&= 2m \qquad\qquad \text{since}\;\;\xi^\dagger\xi = 1
\end{aligned}
$$

3. The momentum space solution of the Dirac equation for the antiparticle is

$$
v^s(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\;\eta \\ -\sqrt{p\cdot\bar{\sigma}}\;\eta \end{pmatrix}
$$

where $\sigma = (I, \sigma)$, $\bar{\sigma} = (I, -\sigma)$ and s labels the spin state $s = 1, 2$. Prove the identity,

$$
\sum_{s=1}^{2} v^s(p)\bar{v}^s(p) = \gamma\cdot p - m
$$

where $\bar{v}^s(p) = v^\dagger(p)\gamma^0$. **[7 Marks]**

Answer 3

$$
\begin{aligned}
\sum_{s=1}^{2} v^s(p)\bar{v}^s(p) &= \sum_{s=1,2} \begin{pmatrix} \sqrt{p\cdot\sigma}\;\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\;\eta^s \end{pmatrix} (\sqrt{p\cdot\sigma}\;\eta^{s\dagger} - \sqrt{p\cdot\bar{\sigma}}\;\eta^{s\dagger}) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
&= \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\;\eta^s \\ -\sqrt{p\cdot\bar{\sigma}}\;\eta^s \end{pmatrix} \begin{pmatrix} -\sqrt{p\cdot\bar{\sigma}}\;\eta^{s\dagger} & \sqrt{p\cdot\sigma}\;\eta^{s\dagger} \end{pmatrix} \\
&= \begin{pmatrix} -\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}}\;\eta^s\eta^{s\dagger} & (p\cdot\sigma)\eta^s\eta^{s\dagger} \\ (p\cdot\bar{\sigma})\eta^s\eta^{s\dagger} & -\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}}\;\eta^s\eta^{s\dagger} \end{pmatrix} \\
&= \begin{pmatrix} -\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} & (p\cdot\sigma) \\ (p\cdot\bar{\sigma}) & -\sqrt{p\cdot\sigma}\sqrt{p\cdot\bar{\sigma}} \end{pmatrix} \sum_s \eta^s\eta^{s\dagger}
\end{aligned}
$$

Now

$$
\sum_s \eta^s\eta^{s\dagger} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
$$

$$\begin{aligned}
\therefore \sum_{s=1}^{2} v^s(p)\bar{v}^s(p) &= \begin{pmatrix} -\sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})} & (p\cdot\sigma) \\ (p\cdot\bar{\sigma}) & -\sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -\sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})} & (p\cdot\sigma) \\ (p\cdot\bar{\sigma}) & -\sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})} \end{pmatrix}
\end{aligned}$$

Now

$$\begin{aligned}
p\cdot\sigma &= p^0\sigma^0 - \mathbf{p}\cdot\sigma \\
p\cdot\bar{\sigma} &= p^0\sigma^0 + \mathbf{p}\cdot\sigma \\
-\sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})} &= -\sqrt{(p^0\sigma^0 - \mathbf{p}\cdot\sigma)(p^0\sigma^0 + \mathbf{p}\cdot\sigma)} \\
&= -\sqrt{(p^0)^2 - |\mathbf{p}|^2} \\
&= -\sqrt{E_{\mathbf{p}}^2 - |\mathbf{p}|^2} \\
&= -m \\
\therefore \sum_{s=1}^{2} v^s(p)\bar{v}^s(p) &= \begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m \end{pmatrix} \\
&= \gamma^0p^0 + \gamma^1p^1 + \gamma^2p^2 + \gamma^3p^3 - m = \gamma\cdot\mathbf{p} - m.
\end{aligned}$$

4. The Dirac equation is given by

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0.$$

Find the adjoint ("barred" version) of the Dirac equation which employs $\bar{\psi}$ instead of ψ. [8 **Marks**]

Answer 4

$$\begin{aligned}
i\gamma^\mu\partial_\mu\psi - m\psi &= 0 \\
i\gamma^0\partial_0\psi + i\gamma^i\partial_i\psi - m\psi &= 0 \\
i\gamma^0(\partial_0\psi^\dagger) - i(-\gamma^i)\partial_i\psi^\dagger - m\psi^\dagger &= 0
\end{aligned}$$

Post multiply by γ^0

$$-i\gamma^0(\partial_0\psi^\dagger)\gamma^0 + i\gamma^i(\partial_i\psi^\dagger)\gamma^0 - m\psi^\dagger\gamma^0 = 0$$

Now $\psi^\dagger\gamma^0 = \bar\psi \quad \Rightarrow \quad \psi^\dagger = \bar\psi\gamma^0$

$$\therefore \quad -i\gamma^0(\partial_0\bar\psi)\gamma^0\gamma^0 + i\gamma^i(\partial_i\bar\psi)\gamma^0\gamma^0 - m\bar\psi\gamma^0\gamma^0 = 0$$

Since $(\gamma^0)^2 = 1$

$$\begin{aligned} -i\gamma^0(\partial_0\bar\psi) + i\gamma^i(\partial_i\bar\psi) - m\bar\psi &= 0 \\ i\gamma^0(\partial_0\bar\psi) - i\gamma^i(\partial_i\bar\psi) + m\bar\psi &= 0 \\ i\gamma^\mu\partial_\mu\bar\psi + m\bar\psi &= 0, \qquad \text{as desired.} \end{aligned}$$

5. Derive explicit representation of the boost matrix $\mathbf{K}$ and show by explicit calculations that $[K^1, K^2] = -iJ^3$. Your starting point are the representations $D(\theta^i)$ and $D(\phi^i)$. Assume we are dealing with vectors (spin= 1). [5 **Marks**]

Answer 5

For a boost along $\hat{x}$,

$$D(\phi^1) = \Lambda(\phi^1) = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cosh\phi^1 & \sinh\phi^1 & 0 & 0 \\ \sinh\phi^1 & \cosh\phi^1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$K^1 = +\frac{1}{i}\left.\frac{\partial D(\phi^1)}{\partial\phi^1}\right|_{\phi^1=0}$$

Now

$$\frac{\partial}{\partial\phi}(\cosh\phi) = \frac{1}{2}\frac{\partial}{\partial\phi}(e^\phi + e^{-\phi}) = \frac{e^\phi - e^{-\phi}}{2} = \sinh\phi$$

$$\frac{\partial}{\partial\phi}(\sinh\phi) = \frac{1}{2}\frac{\partial}{\partial\phi}(e^\phi - e^{-\phi}) = \frac{e^\phi + e^{-\phi}}{2} = \cosh\phi$$

$$\therefore \quad K^1 = \frac{1}{i}\left.\begin{pmatrix} \sinh\phi^1 & \cosh\phi^1 & 0 & 0 \\ \cosh\phi^1 & \sinh\phi^1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right|_{\phi^1=0}$$

$$
\begin{aligned}
&= \frac{1}{i}\begin{pmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = -i\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
K^2 &= +\frac{1}{i}\left.\frac{\partial D(\phi^2)}{\partial \phi^2}\right|_{\phi^2=0} \\
&= \frac{1}{i}\frac{\partial}{\partial \phi^2}\begin{pmatrix} \gamma & 0 & -\gamma v & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma v & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = -i\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

Similarly

$$
\begin{aligned}
K^3 &= -i\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
[K^1, K^2] &= K^1K^2 - K^2K^1 \\
&= +\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= +\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$u(\theta^3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta^3 & -\sin\theta^3 & 0 \\ 0 & \sin\theta^3 & \cos\theta^3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$J^3 = -\frac{1}{i}\left.\frac{\partial u(\theta^3)}{\partial\theta^3}\right|_{\theta^3=0}$$

$$= -\frac{1}{i}\left.\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\sin\theta^3 & -\cos\theta^3 & 0 \\ 0 & \cos\theta^3 & -\sin\theta^3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right|_{\theta^3=0}$$

$$= i\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Hence $[K^1, K^2] = -iJ^3$.

6. I define a boost along the x axis as

$$\Lambda^\mu_\nu(x^1) = \begin{pmatrix} \gamma^1 & \gamma^1 v^1 & 0 & 0 \\ \gamma^1 v^1 & \gamma^1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Boosts along other axes are defined correspondingly.

(a) For infinitesimal boosts, construct the boost matrix $\Lambda^\mu_\nu(x^1, x^2, x^3)$. [3 **Marks**]

(b) Construct the infinitesimal rotation matrix $\Lambda^\mu_\nu(\theta^1, \theta^2, \theta^3)$. [3 **Marks**]

(c) Construct the infinitesimal Lorentz matrix $\Lambda^\mu_\nu(x^1, x^2, x^3; \theta^1, \theta^2, \theta^3)$ and show that it can be written as $\Lambda = 1 + \omega$. What is ω? [3 **Marks**]

(d) Show that $\omega^{\mu\nu} = \omega^\mu_\lambda g^{\lambda\nu}$ and $\omega_{\mu\nu} = g_{\mu\lambda}\omega^\lambda_\nu$ are antisymmetric. [3 **Marks**]

(e) What is the relationship between θ^i, v^i and the terms of ω? [3 **Marks**]

Answer 6

(Thanks to Bilal Azam for this.)

(a) According to given boost, for infinitesimal boost by v^j, we can write

$$x'^0 = x^0 + v_i x^i$$
$$x'^j = x^j + v^j x^0$$

Lorentz transformation gives

$$x'^0 = \Lambda^0{}_\nu x^\nu = \Lambda^0{}_0 x^0 + \Lambda^0{}_i x^i$$
$$x'^0 = x^0 + v_i x^i$$

and

$$x'^j = \Lambda^j{}_\nu x^\nu = \Lambda^j{}_0 x^0 + \Lambda^j{}_k x^k$$
$$x'^j = v^j x^0 + \partial^j_k x^k$$

Thus we get,

$$\Lambda^\mu{}_\nu(x^1, x^2, x^3) = i \begin{pmatrix} 1 & v^1 & v^2 & v^3 \\ v^1 & 1 & 0 & 0 \\ v^2 & 0 & 1 & 0 \\ v^3 & 0 & 0 & 1 \end{pmatrix}$$

(b) Rotation matrix can be written as

$$R^i_j(\theta) = \begin{pmatrix} R^1_1 & R^1_2 & R^1_3 \\ R^2_1 & R^2_2 & R^2_3 \\ R^3_1 & R^3_2 & R^3_3 \end{pmatrix}$$

since rotation does not act on time component, so we can substitute this matrix into $\Lambda^\mu{}_\nu$ to get the desired result.

$$\Lambda^\mu{}_\nu(\theta^1, \theta^2, \theta^3) = i \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \theta^3 & -\theta^2 \\ 0 & -\theta^3 & 1 & \theta^1 \\ 0 & \theta^2 & -\theta^1 & 1 \end{pmatrix}$$

(c) In general, infinitesimal Lorentz transformation is the combination of boots and rotations and is given by:

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$
$$x'^{\mu} = (1 + w^{\mu}{}_{\nu}) x^{\nu}$$

where I is the identity matrix and here

$$\Lambda = I + w$$

i.e. a combination of boosts and rotation. Moreover

$$\begin{aligned}
w^{\mu}{}_{\nu} &= \Lambda^{\mu}{}_{\nu} - 1 \\
w^{\mu}{}_{\nu} &= \begin{pmatrix} 1 & v^1 & v^2 & v^3 \\ v^1 & 1 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 1 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & v^1 & v^2 & v^3 \\ v^1 & 0 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 0 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 0 \end{pmatrix}
\end{aligned}$$

(d)

$$\begin{aligned}
w^{\mu\nu} &= w^{\mu}{}_{\lambda} g^{\lambda\mu} \\
&= \begin{pmatrix} 0 & v^1 & v^2 & v^3 \\ v^1 & 0 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 0 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -v^1 & -v^2 & -v^3 \\ v^1 & 0 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 0 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 0 \end{pmatrix}
\end{aligned}$$

and

$$
\begin{aligned}
w_{\mu\nu} &= g_{\lambda\mu} w^{\lambda}_{\mu} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & v^1 & v^2 & v^3 \\ v^1 & 0 & \theta^3 & -\theta^2 \\ v^2 & -\theta^3 & 0 & \theta^1 \\ v^3 & \theta^2 & -\theta^1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & v^1 & v^2 & v^3 \\ -v^1 & 0 & \theta^3 & -\theta^2 \\ -v^2 & -\theta^3 & 0 & \theta^1 \\ -v^3 & \theta^2 & -\theta^1 & 0 \end{pmatrix}
\end{aligned}
$$

Thus

$$
w_{\mu\nu} = -w^{\mu\nu}
$$

(e)

$$
\begin{aligned}
w^{0i} &= v^i \\
w^{ij} &= \epsilon^{ijk}\theta^k \\
\epsilon^{i\ell m} w^{ij} &= \epsilon^{i\ell m}\epsilon^{ijk}\theta^k \\
&= (\partial^{\ell j}\partial^{mk} - \partial^{\ell k}\partial^{mj})\theta^k \\
&= \partial^{\ell j}\partial^{mk}\theta^k - \partial^{\ell k}\partial^{mj}\theta^k \\
&= \partial^{\ell j}\partial^{m} - \partial^{mj}\theta^{\ell} \\
\partial_{\ell j}\epsilon^{i\ell m} w^{ij} &= \partial_{\ell j}\partial^{\ell j}\partial^{m} - \partial_{\ell j}\partial^{mj}\theta^{\ell} \\
\epsilon^{ijm} w^{ij} &= \partial^{m}_{j}\theta^m - \partial^{m}_{\ell}\theta^{\ell} \\
&= -2\theta^m \\
\theta^m &= -\frac{1}{2}\epsilon^{ijm} w^{ij}.
\end{aligned}
$$