Solution Tutorial 10: Modern Physics

1. Consider a system of two Einstein solids, A and B, each containing 10 oscillators, sharing a total of 10 units of energy. Assume that the solids are brought close and interact, and that the total energy is fixed.
(a) How many different macrostates are available to this system?
(b) How many different microstates are available to this system?
(c) Assuming that this system is in thermal equilibrium, what is the probability of finding all the energy in solid A?
(d) What is the probability of finding exactly half of the energy in solid A?
(e) Under what circumstances would this system exhibit irreversible behavior?

Answer 1:

(a) We are given that,

$$\begin{aligned}
\text{No. of oscillators in solid } A &= N_A = 10 \\
\text{No. of oscillators in solid } B &= N_B = 10 \\
\text{No. of quantums in solid } A &= q_A = 10 \\
\text{No. of quantums in solid } B &= q_B = 10.
\end{aligned}$$

There are **eleven** macrostates when considering the distribution of the 10 **quantums** between the two atoms.
(b) However the number of microstates that can be calculated by the equation of multiplicity of Einstein solid (interacting solids), $\Omega = \frac{(q+N-1)!}{q!(N-1)!}$ is much larger as seen in the Table below.

$\mathbf{q_A}$	0	1	2	3	4	5	6	7	8	9	10
$\mathbf{q_B}$	10	9	8	7	6	5	4	3	2	1	0
$\mathbf{\Omega_A}$	1	10	55	220	715	2002	5005	11440	24310	48620	92378
$\mathbf{\Omega_B}$	92378	48620	24310	11440	5005	2002	715	220	55	10	1
$\mathbf{\Omega_A \Omega_B}$	92378	486200	1337050	2516800	3578575	4008004	3578578	2516800	1337050	486200	92378

The total number of microstates is the sum of entries in the last row of the table

$= 20030010.$

(c) If the system is in thermal equilibrium, then the probability of finding all the energy in solid A is,

$$\begin{aligned}\text{probability} &= \frac{\text{No. of microstates in solid } A}{\text{Total no. of microstates in the system}} \\ &= \frac{92378}{20030010} = 4.6 \times 10^{-3} = 4.6 \times 10^{-1}\% = 0.46\%.\end{aligned}$$

(d) The probability of finding exactly half of the energy in solid A is,

$$\text{probability} = \frac{4008004}{20030010} = 0.2 = 20\%.$$

(e) The system will exhibit nearly irreversible behavior when it starts off from one of the smaller probability macrostates, i.e. when initially, all the energy is stored in either of the objects.

2. Suppose you flip 20 fair coins.

(a) How many possible outcomes (microstates) are there?

(b) What is the probability of getting the sequence HTHHTTTHTHHHTHHHHTHT (in exactly that order)?

(c) What is the probability of getting 12 heads and 8 tails (in any order)?

Answer 2:

(a) For the case of "binary" coins, microstates can be calculated by the equation,

$$\begin{aligned}\Omega(N, n) &= \frac{N!}{q!(N-q)!} \\ \Rightarrow \Omega(20, 0) &= \frac{20!}{0!(20-0)!} = \frac{20!}{0!(20)!} = 1 \\ \Omega(20, 1) &= \frac{20!}{1!(20-1)!} = \frac{20!}{1!(19)!} = 20 \\ \Omega(20, 2) &= \frac{20!}{2!(20-2)!} = \frac{20!}{2!(18)!} = 190,\end{aligned}$$

and so on.

But there is also a simple formula,

$$\sum_{n=0}^{N} \Omega(N, n) = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} = 2^n.$$

We can prove this formula using the binomial theorem,

$$(x+y)^n = \,{}^nC_0 x^n y^0 + \,{}^nC_1 x^{n-1} y^1 + \,{}^nC_2 x^{n-2} y^2 + ... + \,{}^nC_n x^0 y^n.$$

Setting $x = y = 1$ in the above, we obtain,

$$\begin{aligned}(1+1)^n = 2^n &= \,{}^nC_0 + \,{}^nC_1 + \,{}^nC_2 + ... + \,{}^nC_n \\ &= \sum_{k=0}^{n} \,{}^nC_k = \frac{n!}{k!(n-k)!}.\end{aligned}$$

Hence

$$\sum_{n=0}^{20} \Omega(20, n) = 2^{20} = 1,048,576.$$

(b) Probability of getting the sequence HTHHTTTHTHHHTHHHHTHT is,

$$\text{Probability} = \frac{1}{1,048,576} = 9.54 \times 10^{-7}.$$

(c) The number of microstates with exactly 12 heads are,

$$\Omega(20, 12) = \frac{20!}{12!(20-12)!} = \frac{20!}{12!(8)!} = 9690.$$

Hence the probability of obtaining 12 heads is,

$$\text{Probability} = \frac{9690}{1,048,576} = 9.2 \times 10^{-3}.$$

3. (a) A paramagnet has all its dipoles aligned parallel to a magnet field. What is its entropy?

(b) What is the energy of the system in this state? N is the total number of dipoles.

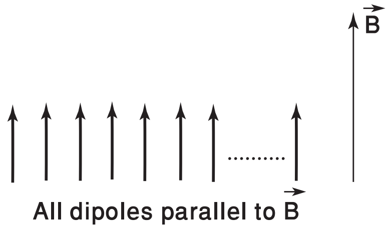
(c) If $N = 10^{23}$, how many microstates are accessible to the system?

(d) If $N = 10^{23}$, and with the huge number of microstates accessible, can the energy still be zero?

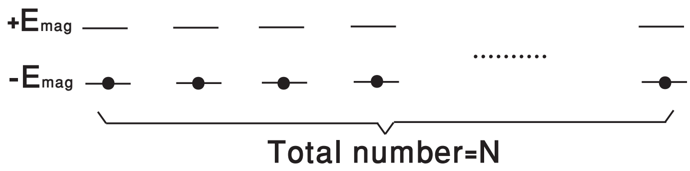
(e) Suppose that the microstates of the system changes a billion times per second. How many microstates will it explore in ten billion years (the age of the universe)? Suppose 1yr $\sim 10^7$ s.

Answer 3:

We are given that, all dipoles in a paramagnet are aligned to the magnetic field, as shown in the figure below.



Hence each dipole has the same energy $-E_{mag}$. The picture in the energy landscape is shown below,



Since all the dipoles are necessarily in the ground state, the macrostate corresponding to perfect alignment has $\Omega = 1$. Hence,

$$S \;=\; k_B \ln \Omega = k_B \ln(1) = 0 \text{ J/K}.$$

(b) If N is the total number of dipoles, then total energy of the system is,

$$E_{\text{Total}} = -N E_{\text{mag}}.$$

(c) If $N = 10^{23}$, microstates with $N_\uparrow$ are,

$$\Omega(N, N_\uparrow) = \Omega(10^{23}, N_\uparrow) = \frac{10^{23}!}{N_\uparrow! 10^{23}!},$$

where $N_\uparrow$ can vary from 1 to N.
The total number of microstates accessible,

$$\sum_{N_\uparrow=0}^{N} \Omega(N, N_\uparrow) = 2^N = 2^{10^{23}},$$

which is a huge number.
(d) Yes if half of the dipoles are in one direction and the other half in the other, $N_\uparrow = N_\downarrow = N/2$. Total energy is,

$$E_{\text{Total}} = -\frac{N}{2} E_{\text{mag}} + \frac{N}{2} E_{\text{mag}} = 0.$$

Total energy is zero but the entropy is maximum.

(d) No. of microstates explored in 10×10^9 years is,

$$\begin{aligned}\text{Microstates explored} &\sim 10 \times 10^9 \text{ years} \times 10^9 \\ &\sim 10 \times 10^9 \times 10^7 \times 10^9 \\ &\sim 10^{26}.\end{aligned}$$

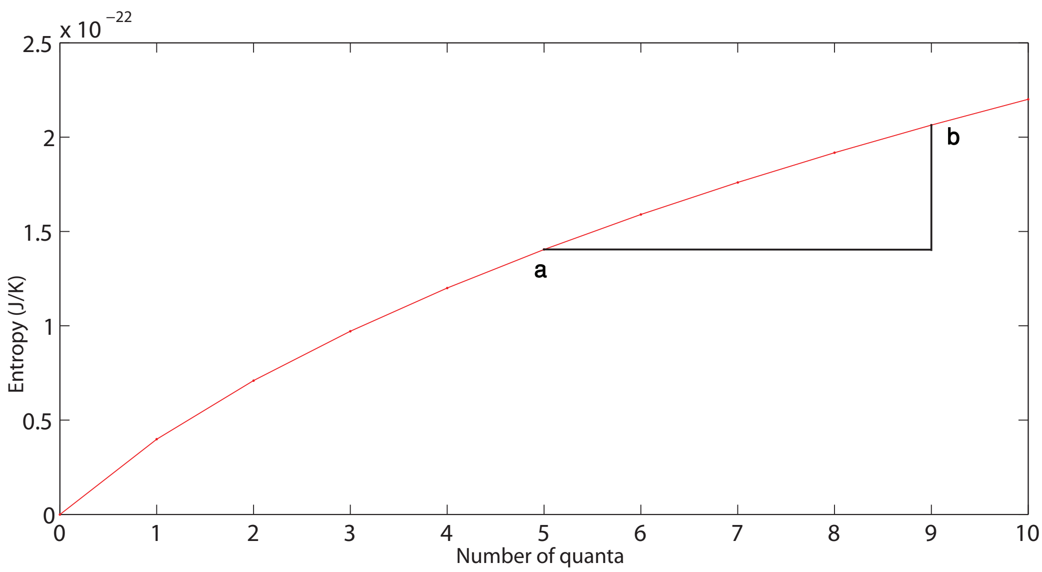
Hence even in its lifetime, the system can't explore all $2^{10^{23}}$ states!

4. A nanoparticle containing 6 atoms can be modeled approximately as an Einstein solid of 18 independent oscillators. The evenly spaced energy levels of each oscillator are 4×10^{-21} J apart.

(a) When the nanoparticle's energy is in the range $5 \times 4 \times 10^{-21}$ J to $6 \times 4 \times 10^{-21}$ J, what is the approximate temperature? (In order to keep precision for calculating the heat capacity, give the result to the nearest tenth of a Kelvin.

(b) When the nanoparticle's energy is in the range $8 \times 4 \times 10^{-21}$ J to $9 \times 4 \times 10^{-21}$ J, what is the approximate temperature? (In order to keep precision for calculating the heat capacity, give the result to the nearest tenth of a degree.)

(c) When the nanoparticle's energy is in the range $5 \times 4 \times 10^{-21}$ J to $9 \times 4 \times 10^{-21}$ J, what is the approximate heat capacity per atom? For your convenience, the entropy-energy graph is also shown.



Answer 4:

We are given that,

$$\begin{aligned} \text{6 atoms in a nanoparticle} &= \text{18 oscillators} \\ \text{Energy of each quanta} &= q = 4 \times 10^{-21}\ \text{J}. \end{aligned}$$

q	$\Omega = \frac{(q+N-1)!}{q!(N-1)!}$	$k_B \ln(\Omega)$
0	1	0
1	18	0.040
2	171	0.071
3	1140	0.097
4	5985	0.120
5	26334	0.141
6	100947	0.159
7	346104	0.176
8	1081575	0.192
9	3124550	0.206
10	8436285	0.220

$$\text{Energy} = q \times 4 \times 10^{-21}\ \text{J}$$

(a) Approximate temperature T when energy is in between 5 and 6 quanta is $\frac{1}{T}$ = slope of the entropy versus energy graph.

At $q = 5$:

To find the slope, we have

$$\begin{aligned} \frac{1}{T} &= \frac{\text{Change in entropy}}{\text{Change in energy}} \\ &= \frac{(0.159 - 0.141) \times 10^{-21}\ \text{J/K}}{(6-5) \times 4 \times 10^{-21}\ \text{J}} \\ &= 4.5 \times 10^{-3}\ 1/\text{K} \\ \Rightarrow T &= 222.2\ \text{K}. \end{aligned}$$

(b)

$$\begin{aligned} \frac{1}{T} &= \frac{\text{Change in entropy}}{\text{Change in energy}} \\ &= \frac{(0.206 - 0.192) \times 10^{-21} \text{ J/K}}{(9-8) \times 4 \times 10^{-21} \text{ J}} \\ &= 4.5 \times 10^{-3} \text{ 1/K} \\ \Rightarrow T &= 285.7 \text{ K.} \end{aligned}$$

(c) The plot of the entropy versus energy (quanta) is shown in the question. To calculate the heat capacity per atom, one needs to find the energy difference between the points a and b and the corresponding change in temperature.

$$\begin{aligned} T_a &\approx 222.2 \text{ K} \\ T_b &\approx 285.7 \text{ K} \\ \Delta T &= T_b - T_a \approx 63.5 \text{ K} \\ \text{Change in energy} &= 4 \times 10^{-21} \text{ J} \\ \Rightarrow C &= \frac{4 \times 4 \times 10^{-21} \text{ J}}{63.5 \text{ K}} \\ &= 4 \times 6.3 \times 10^{-23} \text{ J/K} \\ &= 25.2 \times 10^{-23} \text{ J/K.} \end{aligned}$$

$\boxed{\text{This is per atom.}}$

5. For a certain metal the stiffness of the interatomic bond and the mass of one atom are such that the spacing of the quantum oscillator energy levels is 1.5×10^{-23} J. A nanoparticle of this metal consisting of 10 atoms has a total thermal energy of 18×10^{-23} J. Assume all the internal energy is of the disordered kind.
(a) What is the entropy of this nanoparticle?
(b) The temperature of the nanoparticle is 87 K. Next we add 18×10^{-23} J to the nanoparticle. By how much does the entropy increase?

<u>**Answer 5:**</u>

(a) We are given that,

$$\begin{aligned}
\text{Energy of one quanta} &= q \equiv 1.5 \times 10^{-23}\ \text{J} \\
\text{Number of atoms} &= 10 \\
\text{Number of oscillators} &= 30 \\
\text{Thermal energy} &= 18 \times 10^{-23}\ \text{J}.
\end{aligned}$$

In order to calculate the entropy of nanoparticle, firstly we should calculate number of microstates. Suppose all the energy is in the form of the thermal energy.

$$\begin{aligned}
\text{Number of Quanta} &= \frac{18 \times 10^{-23}\ \text{J}}{1.5 \times 10^{-23}\ \text{J}} \simeq 12 \\
\text{Number of microstates} = \Omega &= \frac{(12+30-1)!}{12!(30-1)!} = \frac{41!}{12!\ 29!}
\end{aligned}$$

$$\Omega = \frac{3.35 \times 10^{49}}{4.79 \times 10^{8} \times 8.84 \times 10^{30}} = 7.91 \times 10^{9}.$$

Therefore entropy of the nanoparticle is,

$$\begin{aligned}
S &= k_B \ln \Omega = 1.38 \times 10^{-23}\ \text{J/K}\ \ln(7.91 \times 10^{9}) \\
&= 1.38 \times 10^{-23}\ \text{J/K} \times 22.79 = 3.14 \times 10^{-22}\ \text{J/K}.
\end{aligned}$$

(b) Now we are givwn that,

$$\begin{aligned}
\text{Temperature of nanoparticle} &= 87\ \text{K} \\
\text{Heat added to the nanoparticle} &= 18 \times 10^{-23}\ \text{J}.
\end{aligned}$$

Slope dS/dE at this temperature is,

$$\frac{dS}{dE} = \frac{1}{T} = \frac{1}{87} = 0.0115\ \text{K}^{-1}.$$

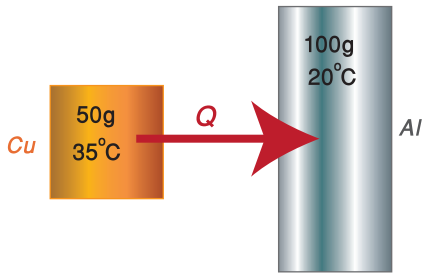
By adding 18×10^{-23} J of energy, entropy goes up by approximately,

$$\text{Change in entropy} = 0.0115\ \text{K}^{-1} \times 18 \times 10^{-23}\ \text{J} = 2.1 \times 10^{-24}\ \text{J/K}.$$

6. A 50 gram block of copper (one mole has a mass of 63.5 grams) at a temperature of 35° C is put in contact with a 100 gram block of aluminum (molar mass 27 grams) at a temperature of 20° C. The blocks are inside an insulated enclosure, with little

contact with the walls. At these temperatures, the high temperature limit is valid for the specific heat capacity, $C_v = 3k_B$. Calculate the final temperature of the two blocks. Do NOT look up the specific heat capacities of aluminum and copper; these have been provided to you.

Answer 6:



We are given that,

$$\begin{aligned}
\text{Mass of copper block} &= 50 \text{ g} \\
\text{Molar mass of copper} &= 63.5 \text{ g} \\
\text{Temperature of copper} &= 35°\text{C} = (35 + 273) \text{ K} = 308 \text{ K} \\
\text{Mass of Aluminium block} &= 100 \text{ g} \\
\text{Molar mass of Aluminium} &= 27 \text{ g} \\
\text{Temperature of Aluminium} &= 20°\text{C} = (20 + 273) \text{ K} = 293 \text{ K} \\
\text{Heat capacity} &= C_v = 3k_B.
\end{aligned}$$

Final temperature of the blocks can be calculated by using the equation,

$$Q = \int_{T_i}^{T_f} C_v dT.$$

Let us first calculate the heat transferred from copper block to Aluminium block.

$$\begin{aligned}
E_{\text{int}}^{\text{Cu}} &= 3k_B T = 3k_B \times 308 \text{ K} = 924 k_B \text{ J/atom} \\
E_{\text{int}}^{\text{Al}} &= 3k_B T = 3k_B \times 293 \text{ K} = 879 k_B \text{ J/atom}.
\end{aligned}$$

Q goes from Cu to Al, because average (or per atom) internal energy in Cu is higher, even though total internal energy in Al is higher because it has more atoms. If Avogadro's number is represented by N_A, then,

$$\begin{aligned}
E_{\rm int}^{\rm Cu}(\text{total}) &= \frac{50 \text{ g}}{63.5 \text{ g/mol}} \times 924k_B \text{ J/atom} \times N_A \text{ atoms} = 727.6k_BN_A \text{ Joules} \\
E_{\rm int}^{\rm Al}(\text{total}) &= \frac{100 \text{ g}}{27 \text{ g/mol}} \times 879k_B \text{ J/atom} \times N_A \text{ atoms} = 3255.6k_BN_A \text{ Joules.}
\end{aligned}$$

A copper atom loses energy xk_B Joules, which is gained by an Al atom. Since energy is conserved,

$$\begin{aligned}
924k_B - xk_B = 879k_B + xk_B & \\
\Rightarrow x &= 22.5.
\end{aligned}$$

Therefore 22.5 Joules of energy are transferred.

$$\begin{aligned}
\Rightarrow \qquad Q &= -22.5k_B \text{ Joules} \\
\text{But} \qquad Q &= \int_{T_i}^{T_f} C_v dT.
\end{aligned}$$

Substituting the value of Q and C_v, we obtained,

$$\begin{aligned}
-22.5k_B &= \int_{T_i}^{T_f} 3k_B dT = 3k_B \int_{T_i}^{T_f} dT \\
&= 3k_B \big|T\big|_{T_i}^{T_f} = 3k_B(T_f - T_i) \\
&= 3k_B(T_f - 308) \\
\Rightarrow T_f &= 301.5 \text{ K} \\
&= 28.5^\circ \text{ C.}
\end{aligned}$$