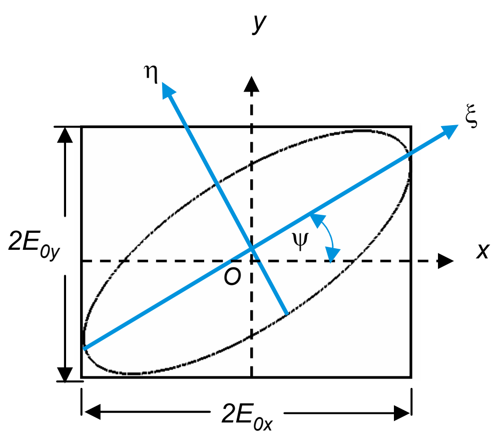
The Polarization Ellipse

According to Fresnel's theory, $E_x(z,t)$ and $E_y(z,t)$ describe sinusoidal oscillations in the x-z and y-z planes, respectively (see the figure on p. 6) By themselves, these equations are not particularly revealing. However, eliminating the time-space propagator $\omega t - kz$ between the two equations leads to the **equation of an ellipse**, namely,

$$\frac{E_x(z,t)^2}{E_{0x}{}^2} + \frac{E_y(z,t)^2}{E_{0y}{}^2} - \frac{2E_x(z,t)E_y(z,t)}{E_{0x}E_{0y}}\cos\delta = \sin^2\delta \,,$$

where $\delta = \delta_y - \delta_x$. The above equation describes an ellipse in its nonstandard form. Because the equation refers to polarized light, the equation is called the **polarization ellipse**. In the equation, the time-space propagator has been explicitly eliminated. Nevertheless, the field components $E_x(z,t)$ and $E_y(z,t)$ continue to be time-space dependent. A plot of the nonstandard polarization ellipse is shown below.

The figure also shows the rotated ξ-η coordinate system. Because of the amplitudes E_{0x} and E_{0y} and the phase δ are constant, the polarization ellipse remains fixed as the polarized beam propagates.



Degenerate Polarization States

In general, the optical field is elliptically polarized, but there are several combinations of amplitude and phase that are especially important. These are called **degenerate polarization states**: (1) linearly horizontal/vertical polarized light (**LHP**/**LVP**), (2) linear ±45° polarized light (L+45P/L–45P), and (3) right/left circularly polarized light (**RCP**/**LCP**). The polarization states along with the mathematical conditions and corresponding figures (polarization ellipses) are as follows.

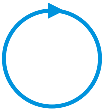
LHP: $E_{0y} = 0$

LVP: $E_{0x} = 0$

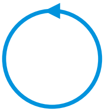
L+45P:
$E_{0x} = E_{0y} = E_0,\ \delta = 0$

L-45P:
$E_{0x} = E_{0y} = E_0,\ \delta = \pi$

RCP:
$E_{0x} = E_{0y} = E_0,\ \delta = \pi/2$



LCP:
$E_{0x} = E_{0y} = E_0,\ \delta = -\pi/2$



RCP light rotates clockwise and LCP rotates counter-clockwise when propagating toward the observer.

These polarization states are important because (1) they are relatively easy to create in a laboratory using linear and **circular polarizers**, and (2) polarization measurements as well as many polarization calculations are greatly simplified using these specific polarization states. This is especially true when a polarized beam propagates through numerous polarizing elements.

The Parameters of the Polarization Ellipse

The polarization ellipse can be expressed in terms of two angular parameters: the **orientation angle** $\psi (0 \le \psi \le \pi)$ and the **ellipticity angle** $\chi (-\pi/4 < \chi \le \pi/4)$.

These angles can be defined in terms of the **parameters of the polarization ellipse**:

$$\tan 2\psi = \frac{2E_{0x}E_{0y}}{E_{0x}^2 - E_{0y}^2}\cos\delta, \qquad 0 \le \psi \le \pi,$$

$$\sin 2\chi = \frac{2E_{0x}E_{0y}\sin\delta}{E_{0x}^2 + E_{0y}^2}, \qquad -\pi/4 < \chi \le \pi/4.$$

The right-hand side of both of these equations consists of algebraic and trigonometric terms. The two equations can be rewritten completely in trigonometric terms by introducing an angle known as the **auxiliary angle** α defined by

$$\tan\alpha = \frac{E_{0y}}{E_{0x}}, \qquad 0 \le \alpha \le \pi/2.$$

This leads to purely trigonometric equations

$$\tan 2\psi = (\tan 2\alpha)\cos\delta,$$

$$\sin 2\chi = (\sin 2\alpha)\sin\delta.$$

The conditions on the angles are $0 \le \alpha \le \pi/2$ and $0 \le \delta < 2\pi$.

E ample We determine the orientation and the ellipticity angles ψ and χ for RCP light. We have for RCP light that $E_{0y} = E_{0x} = E_0$ and $\delta = \pi/2$. Then, tan α yields $\alpha = 45°$ and

$$\tan 2\psi = \tan 90° \cos 90° = 0$$

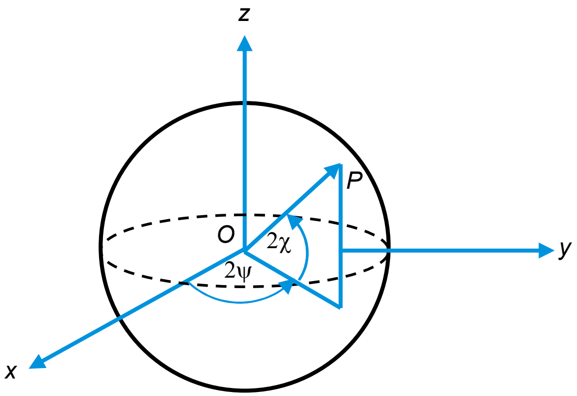
$$\sin 2\chi = \sin 90° \sin 90° = +1$$

Thus, the angles for RCP light are $\psi = 0°$ and $\chi = +45°$.

The Poincaré Sphere

By itself, the polarization ellipse is an excellent way to visualize polarized light. However, except for the degenerate polarization states, it is practically impossible to determine the orientation and ellipticity angles viewing the polarization ellipse. Furthermore, the calculations required to determine the new angles of a polarized beam that propagates through one or more polarizing elements are difficult and tedious.

In order to overcome these difficulties **Poincaré** (1892) suggested using a sphere now known as the **Poincaré sphere** to represent polarized light. The following figure shows the Poincaré sphere and its spherical and Cartesian coordinates.



Here x, y, and z are Cartesian coordinate axes, ψ and χ are the spherical orientation and ellipticity angles (of the polarization ellipse), and P is a point on the surface of the sphere. Note that on the sphere the angles are expressed as 2ψ and 2χ. For a unit sphere the Cartesian coordinates are related to the spherical coordinates by the equation.

$$x = \cos(2\chi)\cos(2\psi),\ 0 \le \psi < \pi,$$

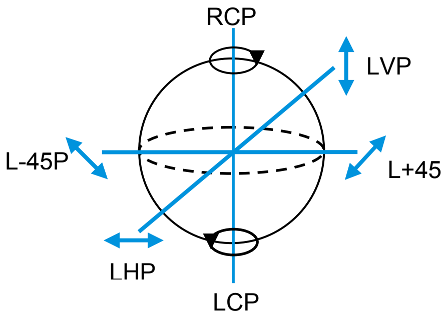
$$y = \cos(2\chi)\sin(2\psi)\,,\ -\pi/4 < \chi \le \pi/4$$

$$z = \sin(2\chi),$$

where $x^2+y^2+z^2=1$ for a sphere of unit radius.

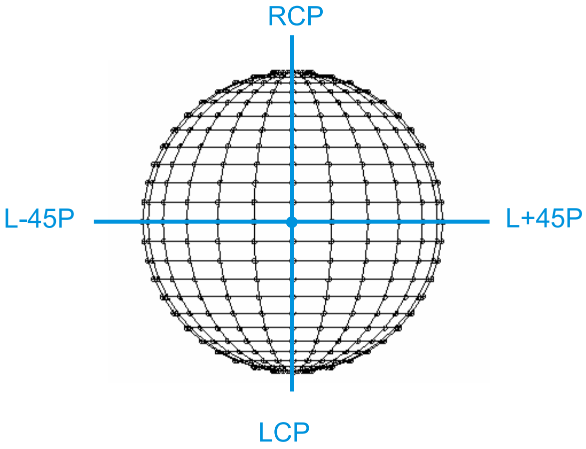
Degenerate States on the Poincaré Sphere

From the previous equations any polarization state can be represented by the coordinate pair (2ψ,2χ). The degenerate polarization states on the Poincaré sphere are for LHP(0°,0°), for L+45P(+90°,0°), for LVP(180°,0°), for L-45P(270°,0°), for RCP(0°,+90°), and for LCP(0°,–90°). The **degenerate states** on the x, y, and z axes are shown below.



All linear polarization states lie on the equator and right and left circular polarization states are at the north and south poles, respectively. Elliptically polarized states are represented everywhere else on the surface of the sphere.

The following figure shows polarization states plotted at every intersection of the 7.5° latitude and 15° longitude lines.



The Stokes Polarization Parameters

The most serious limitation to the Poincaré sphere and the polarization ellipse are (1) the polarization ellipse is an instantaneous representation of polarized light, and (2) neither the rotation angle ψ nor the ellipticity angle χ is directly measurable. In order to overcome these limitations it is necessary to determine the measurables of the polarized field. This can be done by taking a **time average of the polarization ellipse**:

$$\frac{E_x(z,t)^2}{E_{0x}{}^2} + \frac{E_y(z,t)^2}{E_{0y}{}^2} - \frac{2E_x(z,t)E_y(z,t)}{E_{0x}E_{0y}}\cos\delta = \sin^2\delta.$$

The time average $\langle E_i(z,t)E_j(z,t)\rangle$ is defined by

$$\langle E_i(z,t)E_j(z,t)\rangle = \lim_{T\to\infty}\frac{1}{T}\int_0^T E_i(z,t)E_j(z,t)\,dt\,,\qquad i,j = x,y,$$

where T is total averaging time. Applying the time average definition to the polarization ellipse then yields the following equation:

$$S_0{}^2 = S_1{}^2 + S_2{}^2 + S_3{}^2,$$

where

$$S_0 = E_{0x}^2 + E_{0y}^2,$$

$$S_1 = E_{0x}^2 - E_{0y}^2$$

$$S_2 = 2E_{0x}E_{0y}\cos\delta,$$

$$S_3 = 2E_{0x}E_{0y}\sin\delta,\ \delta = \delta_y - \delta_x.$$

The quantities S_0, S_1, S_2, and S_3 are the **observables** of the polarized field. They were introduced by Stokes (1852) and are called the **Stokes polarization parameters**.

The Stokes Polarization Parameters (cont'd)

The first Stokes parameter S_0 describes the total intensity of the optical beam; the second parameter S_1 describes the preponderance of LHP light over LVP light; the third parameter S_2 describes the preponderance of L+45P light over L-45P light and, finally, S_3 describes the preponderance of RCP light over LCP light.

The Stokes parameters can be expressed in **complex notation** (in order to bypass formally the time integration) by suppressing the propagator and writing

$$E_x(t) = E_{0x}\exp(i\delta_x),$$

$$E_y(t) = E_{0y}\exp(i\delta_y).$$

The Stokes parameters are then defined in complex notation by the following equations.

$$S_0 = E_x E_x^* + E_y E_y^*,$$

$$S_1 = E_x E_x^* - E_y E_y^*,$$

$$S_2 = E_x E_y^* + E_y E_x^*,$$

$$S_3 = i\left(E_x E_y^* - E_y E_x^*\right),$$

where $i = \sqrt{-1}$ and * represents the complex conjugate.

It is convenient to arrange the Stokes parameters as a column matrix, which is referred to as the **Stokes vector for elliptically polarized light**:

$$S = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix} = \begin{pmatrix} E_{0x}^2 + E_{0y}^2 \\ E_{0x}^2 - E_{0y}^2 \\ 2E_{0x}E_{0y}\cos\delta \\ 2E_{0x}E_{0y}\sin\delta \end{pmatrix}.$$

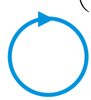
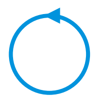
Stokes Parameter Relations

The **Stokes vectors for the degenerate polarization states** are readily found using the previous definitions and equations:

$$S_{\mathrm{LHP}} = I_0 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad S_{\mathrm{LVP}} = I_0 \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, \qquad S_{\mathrm{L+45P}} = I_0 \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$



$$S_{\mathrm{L-45P}} = I_0 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, \qquad S_{\mathrm{RCP}} = I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad S_{\mathrm{LCP}} = I_0 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix},$$

where I_0 is the intensity and is very often normalized to unity.

The Stokes parameters can be shown to be related to the orientation and ellipticity angles, ψ and χ, associated with the Poincaré sphere as follows:

$$S_1 = S_0 \cos(2\chi)\cos(2\psi),$$

$$S_2 = S_0 \cos(2\chi)\sin(2\psi),$$

$$S_3 = S_0 \sin(2\chi),$$

and

$$\psi = \frac{1}{2}\tan^{-1}\left(\frac{S_2}{S_1}\right), \qquad 0 \le \psi \le \pi,$$

$$\chi = \frac{1}{2}\sin^{-1}\left(\frac{S_3}{S_0}\right), \qquad -\frac{\pi}{4} \le \chi \le \frac{\pi}{4}.$$