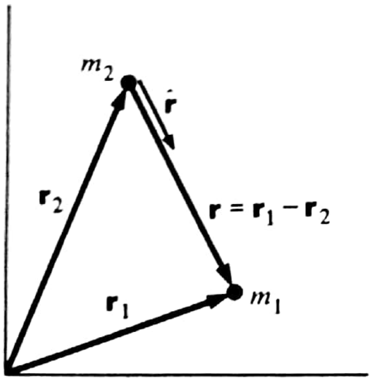
9.1 Introduction

It was Newton's fascination with planetary motion that led him to formulate his laws of motion and the law of universal gravitation. His success in explaining Kepler's empirical laws of planetary motion was an overwhelming argument in favor of the new mechanics and marked the beginning of modern mathematical physics. Planetary motion and the more general problem of motion under a central force continue to play an important role in most branches of physics and turn up in such topics as particle scattering, atomic structure, and space navigation.

In this chapter we apply newtonian physics to the general problem of central force motion. We shall start by looking at some of the general features of a system of two particles interacting with a central force $f(r)\hat{\mathbf{r}}$, where $f(r)$ is any function of the distance r between the particles and $\hat{\mathbf{r}}$ is a unit vector along the line of centers. After making a simple change of coordinates, we shall show how to find a complete solution by using the conservation laws of angular momentum and energy. Finally, we shall apply these results to the case of planetary motion, $f(r) \propto 1/r^2$, and show how they predict Kepler's empirical laws.

9.2 Central Force Motion as a One Body Problem



Consider an isolated system consisting of two particles interacting under a central force $f(r)$. The masses of the particles are m_1 and m_2 and their position vectors are $\mathbf{r}_1$ and $\mathbf{r}_2$. We have

$$\begin{aligned} \mathbf{r} &= \mathbf{r}_1 - \mathbf{r}_2 \\ r &= |\mathbf{r}| \\ &= |\mathbf{r}_1 - \mathbf{r}_2|. \end{aligned} \qquad 9.1$$

The equations of motion are

$$m_1\ddot{\mathbf{r}}_1 = f(r)\hat{\mathbf{r}} \qquad 9.2a$$

$$m_2\ddot{\mathbf{r}}_2 = -f(r)\hat{\mathbf{r}}. \qquad 9.2b$$

The force is attractive for $f(r) < 0$ and repulsive for $f(r) > 0$. Equations (9.2a and b) are coupled together by $\mathbf{r}$; the behavior of $\mathbf{r}_1$ and $\mathbf{r}_2$ depends on $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. We shall show that the problem is easier to handle if we replace $\mathbf{r}_1$ and $\mathbf{r}_2$ by $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and the center of mass vector $\mathbf{R} = (m_1\mathbf{r}_1 + m_2\mathbf{r}_2)/(m_1 + m_2)$. The equation of motion for $\mathbf{R}$ is trivial since there are no external forces. The equation for $\mathbf{r}$ turns out to be like the equation of motion of a single particle and has a straightforward solution.

The equation of motion for $\mathbf{R}$ is

$$\ddot{\mathbf{R}} = 0,$$

which has the simple solution

$$\mathbf{R} = \mathbf{R}_0 + \mathbf{V}t. \tag{9.3}$$

The constant vectors $\mathbf{R}_0$ and $\mathbf{V}$ depend on the choice of coordinate system and the initial conditions. If we are clever enough to take the origin at the center of mass, $\mathbf{R}_0 = 0$ and $\mathbf{V} = 0$.

To find the equation of motion for $\mathbf{r}$ we divide Eq. (9.2a) by m_1 and Eq. (9.2b) by m_2 and subtract. This gives

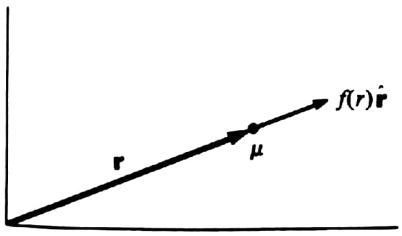
$$\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \left(\frac{1}{m_1} + \frac{1}{m_2}\right) f(r)\hat{\mathbf{r}}$$

or

$$\left(\frac{m_1 m_2}{m_1 + m_2}\right)(\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2) = f(r)\hat{\mathbf{r}}.$$

Denoting $m_1 m_2/(m_1 + m_2)$ by μ, the *reduced mass*, and using $\ddot{\mathbf{r}}_1 - \ddot{\mathbf{r}}_2 = \ddot{\mathbf{r}}$, we have

$$\mu\ddot{\mathbf{r}} = f(r)\hat{\mathbf{r}}. \tag{9.4}$$



Equation (9.4) is identical to the equation of motion for a particle of mass μ acted on by a force $f(r)\hat{\mathbf{r}}$; no trace of the two particle problem remains. The two particle problem has been transformed to a one particle problem. (Unfortunately, the method cannot be generalized. There is no way to reduce the equations of motion for three or more particles to equivalent one body equations, and for this reason the exact solution of the three body problem is unknown.)

The problem now is to find $\mathbf{r}$ as a function of time from Eq. (9.4). Once we know $\mathbf{r}$, we can easily find $\mathbf{r}_1$ and $\mathbf{r}_2$ by using the relations

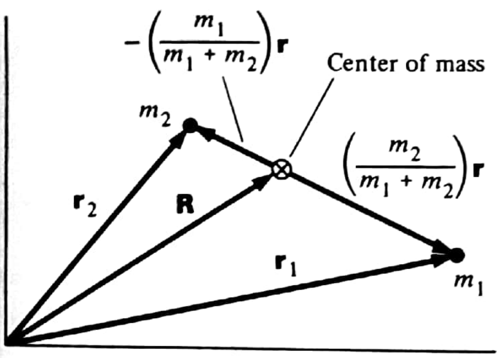
$$\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{9.5a}$$

$$\mathbf{R} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1 + m_2}. \tag{9.5b}$$

Solving for $\mathbf{r}_1$ and $\mathbf{r}_2$ gives

$$\mathbf{r}_1 = \mathbf{R} + \left(\frac{m_2}{m_1 + m_2}\right)\mathbf{r} \tag{9.6a}$$

$$\mathbf{r}_2 = \mathbf{R} - \left(\frac{m_1}{m_1 + m_2}\right)\mathbf{r}. \tag{9.6b}$$



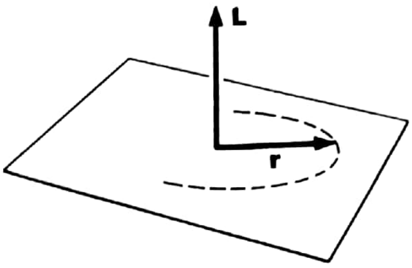
$m_2\mathbf{r}/(m_1 + m_2)$ and $-m_1\mathbf{r}/(m_1 + m_2)$ are the position vectors of m_1 and m_2 relative to the center of mass, as the sketch shows.

The complete solution of $\mu\ddot{\mathbf{r}} = f(r)\,\hat{\mathbf{r}}$ depends on the particular form of $f(r)$. However, a number of the properties of central force motion hold true in general regardless of the form of $f(r)$, and we turn next to investigate these.

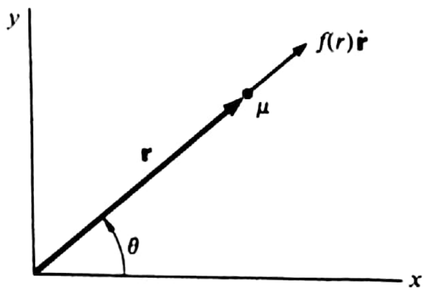
9.3 General Properties of Central Force Motion

The equation $\mu\ddot{\mathbf{r}} = f(r)\,\hat{\mathbf{r}}$ is a vector equation, and although only a single particle is involved, there are three components to be considered. In this section we shall see how to use the conservation laws to find some general properties of the solution and to reduce the equation to an equation in a single scalar variable.

The Motion Is Confined to a Plane



The central force $f(r)\,\hat{\mathbf{r}}$ is along $\mathbf{r}$ and can exert no torque on the reduced mass μ. Hence, the angular momentum $\mathbf{L}$ of μ is constant. It is easy to show that this implies that the motion of μ is confined to a plane. Since $\mathbf{L} = \mathbf{r} \times \mu\mathbf{v}$, where $\mathbf{v} = \dot{\mathbf{r}}$, $\mathbf{r}$ is always perpendicular to $\mathbf{L}$ by the properties of the cross product. However, $\mathbf{L}$ is fixed in space, and it follows that $\mathbf{r}$ can only move in the plane perpendicular to $\mathbf{L}$ through the origin.



Since the motion is confined to a plane, we can, without loss of generality, choose our coordinate system so that the motion is in the xy plane. Introducing polar coordinates, the equation of motion $\mu\ddot{\mathbf{r}} = f(r)\,\hat{\mathbf{r}}$ becomes

$$\mu(\ddot{r} - r\dot{\theta}^2) = f(r) \qquad 9.7a$$

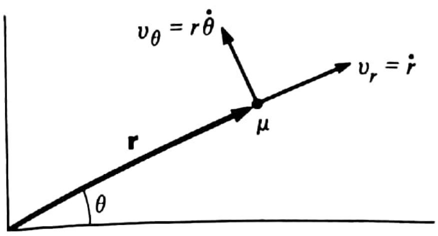
$$\mu(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0. \qquad 9.7b$$

The Energy and Angular Momentum Are Constants of the Motion

We have reduced the problem to two dimensions by using the fact that the direction of $\mathbf{L}$ is constant. There are two other important constants of central force motion: the magnitude of the angular momentum $|\mathbf{L}| \equiv l$, and the total energy E. Using l and E, we can solve the problem of central force motion more easily and with greater physical insight than by working with Eqs. (9.7a and b).

The angular momentum of μ has magnitude

$$l = \mu r v_\theta = \mu r^2\dot{\theta}. \qquad 9.8a$$



The total energy of μ is

$$E = \tfrac{1}{2}\mu v^2 + U(r)$$
$$= \tfrac{1}{2}\mu(\dot r^2 + r^2\dot\theta^2) + U(r), \qquad 9.8b$$

where the potential energy $U(r)$ is given by

$$U(r) - U(r_a) = -\int_{r_a}^{r} f(r)\,dr.$$

The constant $U(r_a)$ is not physically significant and so we can leave r_a unspecified; adding a constant to the energy has no effect on the motion.

We can eliminate $\dot\theta$ from Eq. (9.8b) by using Eq. (9.8a). The result is

$$E = \frac{1}{2}\mu\dot r^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r). \qquad 9.9$$

This looks like the equation of motion of a particle moving in one dimension; all reference to θ is gone. We can press the parallel further by introducing

$$U_{\text{eff}}(r) = \frac{1}{2}\frac{l^2}{\mu r^2} + U(r), \qquad 9.10$$

so that

$$E = \tfrac{1}{2}\mu\dot r^2 + U_{\text{eff}}(r). \qquad 9.11$$

U_{eff} is called the *effective potential energy*. Often it is referred to simply as the *effective potential* v_{eff} differs from the true potential $U(r)$ by the term $l^2/2\mu r^2$, called the *centrifugal potential.*

The formal solution of Eq. (9.11) is

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - U_{\text{eff}})} \qquad 9.12$$

or

$$\int_{r_0}^{r} \frac{dr}{\sqrt{(2/\mu)(E - U_{\text{eff}})}} = t - t_0. \qquad 9.13$$

Equation (9.13) gives us r as a function of t, although the integral may have to be done numerically in some cases. To find θ as a function of t, we can use the solution for r in Eq. (9.8a):

$$\frac{d\theta}{dt} = \frac{l}{\mu r^2}. \qquad 9.14$$

Since r is known as a function of t from Eq. (9.13), it is possible to integrate to find θ:

$$\theta - \theta_0 = \int_{t_0}^{t} \frac{l}{\mu r^2}\,dt. \qquad 9.15$$