Diffraction from a 1D Grating

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Saturday, March, 21, 2023

1 Abstract

This experiment demonstrates diffraction of light from a one-dimensional transmission diffraction grating. The Huygens-Fresnel Principle is used to write Fresnel's diffraction integral formula which after making two approximations, namely Fresnel approximation and Fraunhofer approximation, yields an expression for the intensity of the diffraction pattern; this expression also elucidates the reciprocal relation between the shape of the diffraction pattern and that of the grating that creates it. Using this formula we found the slit separation of the grating to be $h = (3.31 \pm 0.04) \times 10^{-6}$ m with 0.5% error.

2 Introduction

Christian Huygens' principle that asserts a wave front to be made of many wavelets that propagate and interfere with each other to form new wave fronts models light as a wave, but this was overlooked for almost two centuries while Newton's corpuscular theory of light prevailed. Until 1807 when Thomas Young demonstrated the interference of light in his famous double-slit experiment: a phenomenon characteristic of waves. Even then, Young's conclusions received acceptance only gradually, with the exception of Augustin Fresnel.

Fresnel's mathematical prowess allowed him to combine physical insight with mathematical rigor. He adapted Huygens' principle into a formula now called Fresnel's diffraction integral. In light of this, diffraction can be understood as the spilling of wavelets around obstructions in the path of light.

Fresnel's diffraction integral formula is hard to solve analytically, so Fresnel made an approximation to his own formula called the Fresnel approximation. It allows us to describe the electric field of light along its propagation direction after passing

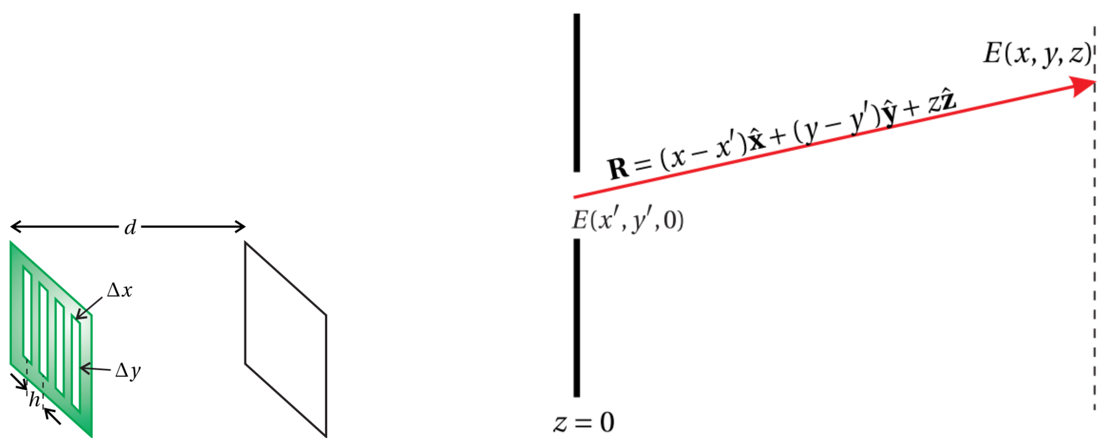
through an aperture. The diffraction pattern changes as the distance from an aperture increases: but at far enough distances, it stops changing, other than to grow in proportion to distance. We can simply the Fresnel integral even further in this far-field limit; this is called the Fraunhofer approximation.

A transmission diffraction grating is an array of equally spaced identical slits. A Reflection diffraction grating is an array of equally spaced identical, rectangular mirrors. We study diffraction from a one-dimensional transmission grating Figure (1a).

This experiment can be used either to find the slit separation of the grating knowing wavelength of the light or to measure the wavelength of the light producing the diffraction pattern while knowing the slit separation of the grating: this is the basis of a diffraction grating spectrometer. Although, a spectrometer has poor resolution compared to a Fabry-Perot interferometer, nevertheless, it can measure a wide range of wavelengths simultaneously.

3 Theoretical background [1]

We motivate the Fresnel's diffraction integral and after applying the Fresnel approximation and the Fraunhofer approximation we arrive at an integral that we can easily evaluate. We find this integral for a rectangular aperture and then use the array theorem to find the intensity profile after passing through a 1D transmission grating.



(a) (b)

Figure 1: (a) A transmission grating and (b) the field at an aperture.

3.1 Fresnel diffraction integral

Each wavelet is the source of a spherical wave proportional to e^{ikR}/R. Consider an aperture in a screen at $z = 0$, Figure (1b). Let it be illuminated with a light having $E(x', y', z = 0)$ within the aperture. For a point (x, y, z) to the right of

the aperture the net field is given by adding together spherical wavelets emitted from each point in the aperture. These spherical wavelets must all have the same strength and phase as the original field. This summation becomes

$$E(x, y, z) = -\frac{i}{\lambda} \iint\limits_{aperture} E(x', y', 0) \frac{e^{ikR}}{R} \mathrm{d}x' \mathrm{d}y', \qquad (1)$$

where

$$R = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}, \qquad (2)$$

is the radius of each wavelet as it crosses the point (x, y, z). The factor $-i/\lambda$ ensures the right phase and field strength as well as units. Equation (1) is the Fresnel diffraction integral.

3.2 Fresnel approximation

Fresnel made an approximation to his formula that requires considering only small angle rays with respect to the z-axis emerging from the aperture. This allows the R to be replaced by z in the denominator of (1) which means we can pull it out of the integral since it no longer depends on x' and y'. Note that we cannot make the same approximation in the numerator of (1) since e^{ikR} is dramatically sensitive to small variations in R. To approximate R in the exponent we expand (3) under the assumption $z^2 \gg (x - x')^2 + (y - y')^2$ which is consistent with the small-angle approximation. This gives

$$R = z\sqrt{1 + \frac{(x - x')^2 + (y - y')^2}{z^2}} \cong z \left[1 + \frac{(x - x')^2 + (y - y')^2}{2z^2} + \ldots \right], \qquad (3)$$

Making these two approximations in (1) yield

$$E(x, y, z) \cong -\frac{ie^{ikz} e^{i\frac{k}{2z}(x^2 + y^2)}}{\lambda z} \iint\limits_{aperture} E(x', y', 0) e^{i\frac{k}{2z}(x'^2 + y'^2)} e^{-i\frac{k}{z}(xx' + yy')} \mathrm{d}x' \mathrm{d}y' \qquad (4)$$

3.3 Fraunhofer approximation

As (4) shows that the field evolves as it propagates along the z-direction, eventually it stops changing and only scales in size (we'll see this in a moment). We are interested in this so called *far-field* limit. This is called the Fraunhofer approximation.

If we look at the $e^{i\frac{k}{2z}(x'^2 + y'^2)}$ factor in (4) we see that if

$$z \gg \frac{k}{2} (\text{aperture radius})^2 \qquad (5)$$

then we can set

$$e^{i\frac{k}{2z}(x'^2 + y'^2)} \approx 1 \qquad (6)$$

in (4) and get

$$E(x,y,z) \cong -\frac{ie^{ikz}e^{i\frac{k}{2z}(x^2+y^2)}}{\lambda z}\iint\limits_{aperture} E(x',y',0)e^{-i\frac{k}{z}(xx'+yy')}\mathrm{d}x'\mathrm{d}y' \tag{7}$$

Since z appears only in the combinations x/z and y/z inside the integral we see that the field can only scale in size and not change its shape as it travels in the z-direction.
Note that this integral can be interpreted as a two-dimensional Fourier transform of $E(x',y',0)$ which explains the reciprocal relation of the diffraction pattern that we will later see and the grating shape that produces it.

3.4 The Array Theorem

The array theorem helps us to find the Fraunhofer diffraction from an array of N identical apertures.

For N apertures in a mask (Figure 2) assume identical field distribution $E(x',y',0)$ for each of them and that this field is zero outside the apertures: this allows us not to worry about the limits of integration in (7) as we can integrate over the entire mask. The mask is considered to be in the xy-plane so that the n^{th} aperture's position is (x'_n, y'_n).

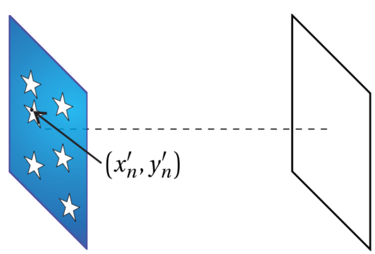


Figure 2: Array of identical apertures

The array theorem states that the electric field of light after passing through an array of N identical apertures (of any shape) is

$$\begin{aligned} E(x,y,z) = &\left[\sum_{n=1}^{N} e^{-i\frac{k}{z}(xx'_n+yy'_n)}\right] \\ &\times \left[-i\frac{e^{ikz}e^{i\frac{k}{2z}(x^2+y^2)}}{\lambda z}\int\limits_{-\infty}^{\infty}\mathrm{d}x'\int\limits_{-\infty}^{\infty}\mathrm{d}y' E_{aperture}(x',y',0)e^{-i\frac{k}{z}(xx'+yy')}\right] \end{aligned} \tag{8}$$

The first factor in brackets contains information about the positions of the identical apertures. The second factor in brackets is the Fraunhofer diffraction pattern from a single aperture centered on $x'=0$ and $y'=0$.

3.5 Diffraction Grating

The array theorem can be used to find the field, and hence intensity, of light after it passes through a one-dimensional transmission grating (Figure 1a) with rectangular slits.

The Fraunhofer diffraction pattern from a single rectangular aperture is

$$E_{aperture}(x,y,z) = -iE_0 \frac{\Delta x \Delta y e^{ikz}}{\lambda z} e^{i\frac{k}{2z}(x^2+y^2)} \operatorname{sinc}\left(\frac{\pi \Delta x}{\lambda z}x\right) \operatorname{sinc}\left(\frac{\pi \Delta y}{\lambda z}y\right) \tag{9}$$

This is the second factor in (8). For the first factor, let the apertures be positioned at

$$x'_n = \left(n - \frac{N+1}{2}\right)h, \qquad y'_n = 0 \tag{10}$$

where N is the total number of slits. The summation in the array theorem (8) becomes

$$\sum_{n=1}^{N} e^{-i\frac{k}{z}(xx'_n + yy'_n)} = e^{i\frac{khx}{z}\left(\frac{N+1}{2}\right)} \sum_{n=1}^{N} e^{-i\frac{khx}{z}n} \tag{11}$$

This is a geometric sum so we get

$$\sum_{n=1}^{N} e^{-i\frac{k}{z}(xx'_n + yy'_n)} = e^{i\frac{khx}{z}\left(\frac{N+1}{2}\right)} e^{-i\frac{khx}{z}} \frac{e^{-i\frac{khx}{z}N} - 1}{e^{-i\frac{khx}{z}} - 1} \tag{12}$$

$$= \frac{e^{-i\frac{khx}{2z}N} - e^{i\frac{khx}{2z}N}}{e^{-i\frac{khx}{2z}} - e^{i\frac{khx}{2z}}} = \frac{\sin\left(N\frac{khx}{2z}\right)}{\sin\left(\frac{khx}{2z}\right)} \tag{13}$$

Substituting (9) and (13) in (8) gives the Fraunhofer diffraction pattern for a diffraction grating:

$$E(x,y,z) = \frac{\sin\left(N\frac{khx}{2z}\right)}{\sin\left(\frac{khx}{2z}\right)} \left[-iE_0 \frac{\Delta x \Delta y e^{ikz}}{\lambda z} e^{i\frac{k}{2z}(x^2+y^2)} \operatorname{sinc}\left(\frac{\pi \Delta x}{\lambda z}x\right) \operatorname{sinc}\left(\frac{\pi \Delta y}{\lambda z}y\right)\right] \tag{14}$$

Now we suppose that the slits are tall enough to satisfy $\Delta y \gg \lambda$ and if the slits are infinitely tall, the final sinc function in (14) becomes one. The intensity pattern (Figure 3) in the horizontal direction becomes

$$I(x) = I_{peak} \operatorname{sinc}^2\left(\frac{\pi \Delta x}{\lambda z}x\right) \frac{\sin^2\left(N\frac{\pi hx}{\lambda z}\right)}{N^2 \sin^2\left(\frac{\pi hx}{\lambda z}\right)} \tag{15}$$

Since $\lim_{\alpha \to 0} \frac{\sin N\alpha}{\sin \alpha} = N$ we have introduced N^2 in the denominator so that the definition of I_{peak} is independent of N and is the intensity at $x = 0$. The diffraction peaks occur when

$$\frac{\pi hx}{\lambda z} = m\pi \qquad m = 0, \pm 1, \pm 2, ... \tag{16}$$

$$x_m = \frac{m\lambda z}{h} \tag{17}$$

This means that we can find h if we know λ and vice versa.

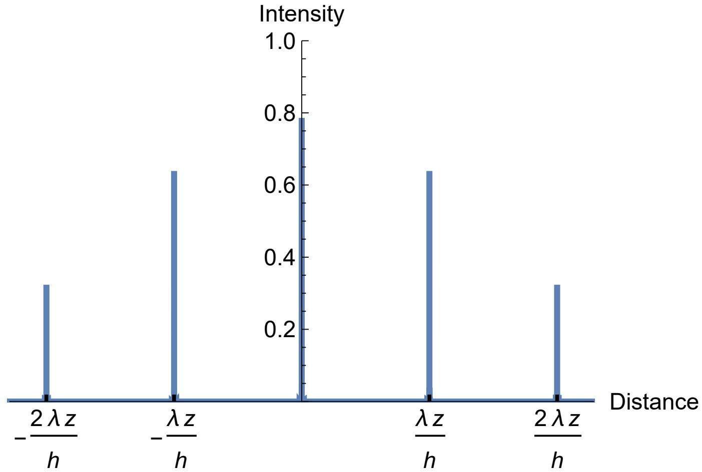


Figure 3: Normalized Intensity profile for a grating with $N = 3810$, $h = 10^{-5}/3$ m, $\Delta x = h/4$ m, $\lambda = 633$ nm at $z = 0.23$ m.

4 Experimental procedure [2]

A 633 nm laser beam from a He-Ne laser is incident on a diffraction grating which has 300 grooves/mm. A paper screen is placed at ≈ 23 cm from the grating and is used to observe the resulting diffraction pattern. Pictures of the diffraction pattern are taken for further analysis.

The setup is arranged as shown in Figure (4). Turning on the laser and making sure that the laser beam travels straight through the grating and falls on the paper screen we should see a diffraction pattern as shown in Figure (5a).

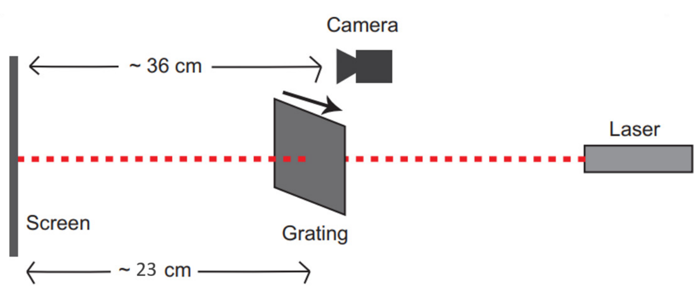
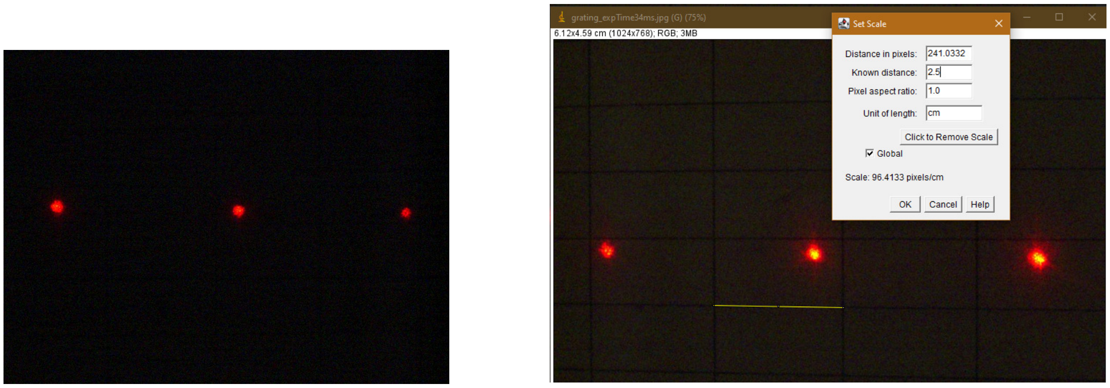


Figure 4: Schematic of the experimental setup. The red line represents the conceived path of the laser beam. The arrow shows orientation of the grating.

The camera is connected to a computer and its accompanying software `uc480 Viewer` is used for operating it. The image viewed by the camera can be displayed live on the computer by navigating to `Live video` $\rightarrow$ `Open camera`. Note that in Figure (3) we have symmetric peaks about the central most intense peak: we see similar symmetry in the peaks viewed by the camera. We can visually determine the central peak since it is the brightest one, and then we position the camera to view the central peak and two other peaks to its right as can be seen in Figure (5a). The two knobs on the camera can be used to adjust its intensity and focus to

get a clear image. If the background of the image is not completely black then adjusting the exposure time by navigating to `uc480` → `Properties` → `Camera` → `Exposure time` can darken it: setting it to a low value (like 5 ms) can blacken the background. An image of the diffraction pattern is saved in `.jpeg` format.



(a) Diffraction pattern with 5 ms exposure time.

(b) Diffraction pattern with 34 ms exposure time and the pixel-to-cm conversion window.

Figure 5

The conversion of this `.jpeg` image into an intensity graph is done using `ImageJ`. Opening this image in `ImageJ` and using two known points in the image (e.g. grid lines on the paper screen) can give us the pixel-to-cm conversion factor (e.g. 241 pixels/cm), see Figure (5b). To obtain the intensity plot we need to select the area containing the diffraction peaks and then click on `Analyze` and choose `Plot Profile`. This will produce an intensity plot with x-axis in cm and y-axis in greyscale magnitude. We then save this data from `ImageJ` in `csv` format for further analysis in a software of our choice (the below are done in `Mathematica`).

5 Results

The purpose of this experiment is to determine the slit separation of the diffraction grating by measuring the positions of the diffraction peaks.

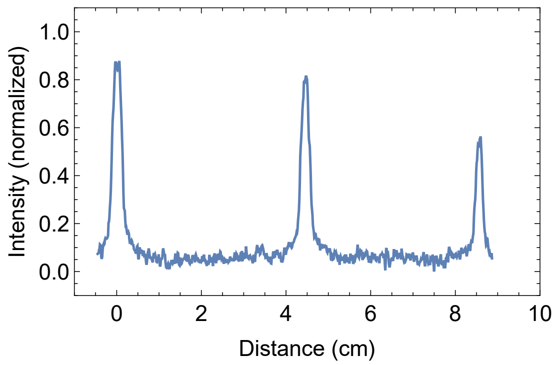
5.1 Slit Separation (Analytical)

The grating used in this experiment has 300 grooves/mm which means that the slit separation is

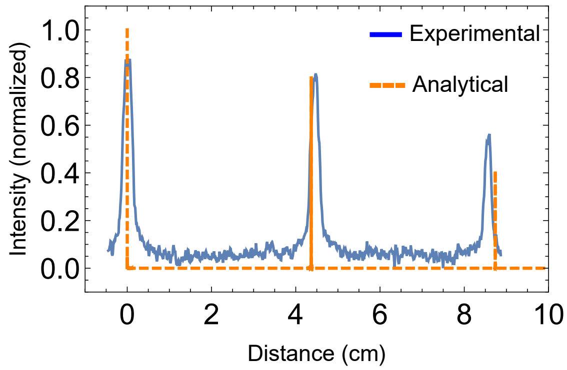
$$h = \frac{1}{n} = \frac{1}{300}\text{mm} = \frac{10^{-5}}{3}\text{m} \tag{18}$$

5.2 Slit Separation (Experimental)

The experimentally obtained intensity grayscale data from `ImageJ` is plotted in Figure (6a).



(a) Experimentally obtained intensity profile.



(b) Experimentally obtained intensity profile fitted with the analytical formula with $\Delta x = h/4$.

Figure 6

We need to fit the analytical expression (15) to this experimental data: the only parameter is the slit-width Δx. Choosing this to be $\Delta x = h/4$ gives the best fit, Figure (6b).

We can find the position of the first peak from the `csv` file obtained from `ImageJ`. It turns out to be $x_1 = 0.0439103$ m. We use this to find the slit separation from (16)

$$h = \frac{\lambda z}{x_1} = \frac{633 \times 10^{-9} \times 0.23}{0.0439103} \tag{19}$$

$$h = 3.31562 \times 10^{-6}\text{m} \tag{20}$$

Looking at the expression for h we see that it has an uncertainty only due to the uncertainties of z and x_1 both of which were measured with a ruler with 1 mm least count making their uncertainties to be ± 0.5 mm. So the uncertainty in h is

$$\Delta h = |h|\sqrt{\left(\frac{\Delta z}{z}\right)^2 + \left(\frac{\Delta x}{x}\right)^2} \tag{21}$$

$$\Delta h = (3.31562 \times 10^{-6})\sqrt{\left(\frac{0.0005}{0.23}\right)^2 + \left(\frac{0.0005}{0.043}\right)^2} \tag{22}$$

$$\Delta h = 3.92217 \times 10^{-8} \tag{23}$$

So the value of h is

$$\boxed{h = (3.31 \pm 0.04) \times 10^{-6}\text{ m}} \tag{24}$$

This has a percentage relative error of 0.53%.

6 Conclusions and discussion

As was mentioned earlier, we can use this experiment in two ways: knowing the wavelength of the laser we can find the slit separation of the grating or the other way around. We have found the slit separation using the wavelength, now let us calculate the wavelength of the laser using the exact value of the slit separation.

$$\lambda = \frac{hx_1}{z} \tag{25}$$

$$\lambda = \frac{0.0439103 \times 10^{-5}}{3 \times 0.23} \tag{26}$$

$$\lambda = 636.4 \times 10^{-9} \tag{27}$$

This too has a relative error of 0.5%. Such accurate measurement of the wavelength of light makes (a variant of) this experiment viable to be used as a diffraction spectrometer.

This also justifies the assumptions we had made in deriving the diffraction intensity equation, (15):

- Each point in a wavelet acts as a source of a spherical wave proportional to e^{ikr}/r.
- The screen is far enough for us to consider only the rays making small angles with the z-axis.
- The field eventually stops evolving and acquires a shape that only scales in size as it travels along the z-axis.
- The slits are significantly larger than the wavelength of the light so that we can take $\Delta y \to \infty$.

References

[1] J. Peatross and M. Ware, "*Physics of Light and Optics*", Brigham Young University, (2011c edition), ch. 10-11.

[2] M. Iqbal, M. Waseem A. Hussain, and M. Anwar, "*Diffraction from One Dimensional Grating*", physlab.org, $v2$ (2018).