2 Pendulums somewhat simple

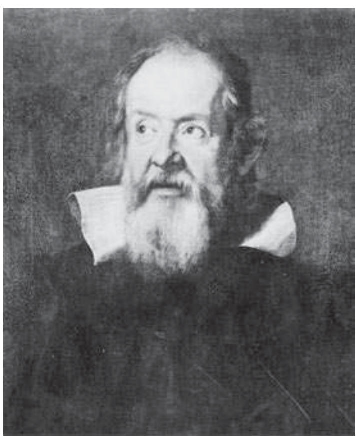

Fig. 2.1
Portrait of Galileo. ©Bettmann/Corbis/Magma.

There are many kinds of pendulums. In this chapter, however, we introduce a simplified model; the small amplitude, linearized pendulum. For the present, we ignore friction and in doing so obviate the need for energizing the pendulum through some forcing mechanism. Our initial discussion will therefore assume that the pendulum's swing is relatively small; and this approximation allows us to linearize the equations and readily determine the motion through solution of simplified model equations. We begin with a little history.

2.1 The beginning

Probably no one knows when pendulums first impinged upon the human consciousness. Undoubtedly they were objects of interest and decoration after humankind learnt to attend routinely to more basic needs. We often associate the first scientific observations of the pendulum with Galileo Galilei (1554–1642; Fig. 2.1).

Fig. 2.2
Cathedral at Pisa. The thin vertical wire indicates a hanging chandelier.

According to the usual story (perhaps apocryphal), Galileo, in the cathedral at Pisa, Fig. 2.2 observed a lamplighter push one of the swaying pendular chandeliers. His earliest biographer Viviani suggests that Galileo then timed the swings with his pulse and concluded that, even as the amplitude of the swings diminished, the time of each swing was constant. This is the origin of Galileo's apparent discovery of the approximate isochronism of the pendulum's motion. According to Viviani these observations were made in 1583, but the Galileo scholar Stillman Drake (Drake 1978) tells us that guides at the cathedral refer visitors to a certain lamp which they describe as "Galileo's lamp," a lamp that was not actually installed until late in 1587. However, there were undoubtedly earlier swaying lamps. Drake surmises that Galileo actually came to the insight about isochronism in connection with his father's musical instruments and then later, perhaps 1588, associated isochronism with his earlier pendulum observations in the cathedral. However, Galileo did make systematic observations of pendulums in 1602. These observations confirmed only approximately his earlier conclusion of isochronism of swings of differing amplitude. Erlichson (1999) has argued that, despite the nontrivial empirical evidence to the contrary, Galileo clung to his earlier conclusion, in part, because he believed that the universe had been ordered so that motion would be simple and that there was "no reason" for the longer path to take a longer time than the shorter path. While Galileo's most famous conclusion about the pendulum has only partial legitimacy, its importance resides (a) in it being the first known scientific deduction about the pendulum, and (b) in the fact that the insight of approximate isochronism is part of the opus of a very famous seminal character in the history of physical science. In these circumstances, the pendulum begins its history as a significant model in physical science and, as we will see, continues to justify its importance in science and technology during the succeeding centuries.

2.2 The simple pendulum

The simple pendulum is an idealization of a real pendulum. It consists of a point mass, m, attached to an infinitely light rigid rod of length l that is itself attached to a frictionless pivot point. See Fig. 2.3. If displaced from its vertical equilibrium position, this idealized pendulum will oscillate with a constant amplitude forever. There is no damping of the motion from friction at the pivot or from air molecules impinging on the rod. Newton's second law, mass times acceleration equals force, provides the equation of motion:

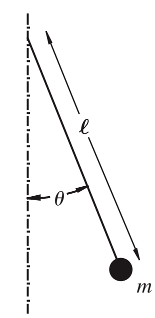

Fig. 2.3
The simple pendulum with a point mass bob.

$$ml\,\frac{d^2\theta}{dt^2} = -mg\sin\theta, \tag{2.1}$$

where θ is the angular displacement of the pendulum from the vertical position and g is the acceleration due to gravity. Equation (2.1) may be simplified if we assume that amplitude of oscillation is small and that $\sin\theta \approx \theta$. We use this *linearization* approximation throughout this chapter. The modified equation of motion is

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0. \tag{2.2}$$

The solution to Eq. (2.2) may be written as

$$\theta = \theta_0 \sin(\omega t + \phi_0), \tag{2.3}$$

where θ_0 is the angular amplitude of the swing,

$$\omega = \sqrt{\frac{g}{l}} \tag{2.4}$$

is the angular frequency, and ϕ_0 is the initial phase angle whose value depends on how the pendulum was started—its initial conditions. The period of the motion, *in this linearized approximation*, is given by

$$T = 2\pi\sqrt{\frac{l}{g}}, \tag{2.5}$$

which is a constant for a given pendulum, and therefore lends support to Galileo's conclusion of isochronism.

The dependence of the period on the geometry of the pendulum and the strength of gravity has very interesting consequences which we will explore. But for the moment we consider further some of the mathematical relationships. Figure 2.4 shows the angular displacement $\theta = \theta_0 \sin(\omega t + \phi_0)$ and the angular velocity $\dot{\theta} = \theta_0 \omega \cos(\omega t + \phi_0)$, respectively, as functions of time. We refer to such graphs as time series. The displacement and velocity are 90 degrees out of phase with each other and therefore when one quantity has a maximum absolute value the other quantity is zero. For example, at the bottom of its motion the pendulum has no angular displacement yet its velocity is greatest.

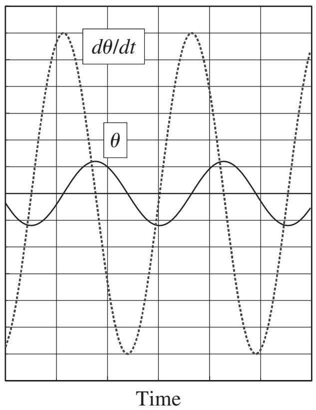

Fig. 2.4
Time series for the angular displacement θ and the angular velocity, $\dot{\theta}$.

The relationship between angle and velocity may be represented graphically with a *phase plane diagram*. In Fig. 2.5 angle is plotted on the horizontal axis and angular velocity is plotted on the vertical axis. As time goes on, a point on the graph travels around the elliptically shaped curve. In effect, the equations for angle and angular velocity are considered to be parametric equations for which the parameter is proportional to time. Then the *orbit* of the *phase trajectory* is the ellipse

$$\frac{\theta^2}{\theta_0^2} + \frac{\dot{\theta}^2}{(\omega\theta_0)^2} = 1. \tag{2.6}$$

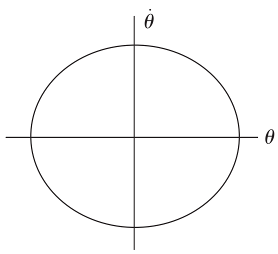

Fig. 2.5
Phase plane diagram. As time increases the phase point travels around the ellipse.

Since the motion has no friction nor any forcing, energy is conserved on this phase trajectory. Therefore the sum of the kinetic and potential energies at any time can be shown to be constant as follows. In the linearized approximation,

$$E = \frac{1}{2} ml^2\dot{\theta}^2 + \frac{1}{2} mgl\theta^2 \tag{2.7}$$

and, using Eqs. (2.3) and (2.4), we find that

$$E = \frac{1}{2} mgl\theta_0^2, \tag{2.8}$$

which is the energy at maximum displacement.

The *phase plane* is a useful tool for the display of the dynamical properties of many physical systems. The linearized pendulum is probably one of the simplest such systems but even here the phase plane graphic is helpful. For example, Eq. (2.6) shows that the axes of the ellipse in Fig. 2.5 are determined by the amplitude and therefore the energy of the motion. A pendulum of smaller energy than that shown would exhibit an ellipse that sits inside the ellipse of the pendulum of higher energy. See Fig. 2.6. Furthermore the two ellipses would never intersect because such intersection implies that a pendulum can jump from one energy to another without the agency of additional energy input. This result leads to a more general conclusion called the *no-crossing theorem*; namely, that orbits in phase space never cross. See Fig. 2.7.

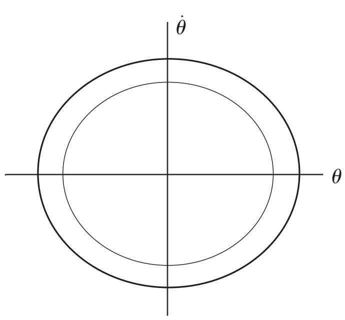

Fig. 2.6
Phase orbits for pendulums with different energies, E_1 and E_2.

Why should this be so? Every orbit is the result of a deterministic equation of motion. Determinism implies that the orbit is well defined and that there would be no circumstance in which a well determined particle would arrive at some sort of ambiguous junction point where its path would be in doubt. (Later in the book we will see *apparent* crossing points but these false crossings are the result of the system arriving at the same phase coordinates at *different* times.)

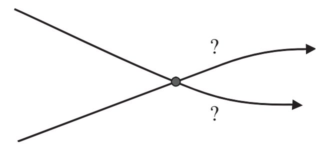

Fig. 2.7
If two orbits in phase space intersect, then it is uncertain which orbit takes which path from the intersection. This uncertainty violates the deterministic basis of classical mechanics.

We introduce one last result about orbits in the phase plane. In Fig. 2.6 there are phase trajectories for two pendulums of different energy. Now think of a large collection of pendulums with energies that are between the two trajectories such that they have very similar, but not identical, angles and velocities. This cluster of pendulums is represented by a set of many *phase points* such that they appear in the diagram as an approximately solid block between the original two trajectories. As the group of pendulums executes their individual motions the set of phase points will move between the two ellipses in such a way that the area defined by the boundaries of the set of points is preserved. This preservation of *phase area*, known as Liouville's theorem (after Joseph Liouville (1809–1882)) is a consequence of the conservation of energy property for each pendulum. In the next chapter we will demonstrate how such areas decrease when energy is lost in the pendulums. But for now let us show how phase area conservation is true for the very simple case when $\theta_0 = 1, \phi = 0$, and $\omega = 1$. In this special case, the ellipses becomes circles since the axes are now equal. See Fig. 2.8. A block of points between the circles is bounded by a small polar angle interval $\Delta\alpha$, in the phase space, that is proportional to time. Each point in this block rotates at the same rate as the motion of its corresponding pendulum progresses. Therefore, after a certain time, all points in the original block have rotated, by the same polar angle, to new positions again bounded by the two circles. Clearly, the size of the block has not changed, as we predicted.

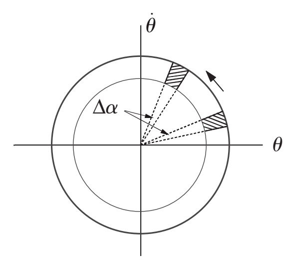

Fig. 2.8
Preservation of area for conservative systems. A block of phase points keeps its same area as time advances.

The motion of the pendulum is an obvious demonstration of the alternating transformation of kinetic energy into potential energy and the reverse. This phenomenon is ubiquitous in physical systems and is known as *resonance*. The pendulum resonates between the two states (Miles 1988*b*). Electrical circuits in televisions and other electronic devices resonate. The terms *resonate* and *resonance* may also refer to a sympathy between two or more physical systems, but for now we simply think of resonance as the periodic swapping of energy between two possible formats.

We conclude this section with the introduction of one more mathematical device. Its use for the simple pendulum is hardly necessary but it will be increasingly important for other parts of the book. Almost two hundred years ago, the French mathematician Jean Baptiste Fourier (1768–1830) showed that periodic motion, whether that of a simple sine wave like our pendulum, or more complex forms such as the triangular wave that characterizes the horizontal sweep on a television tube, are simple linear sums of sine and cosine waves now known as *Fourier Series*. That is, let $f(t)$

be a periodic function such that $f(t) = f(t + (2\pi)/\omega_0)$, where $T = (2\pi)/\omega_0$ is the basic periodicity of the motion. Then Fourier's theorem says that this function can be expanded as

$$f(t) = \sum_{n=1}^{\infty} b_n \cos n\omega_0 t + \sum_{n=1}^{\infty} c_n \sin n\omega_0 t + d, \tag{2.9}$$

where the coefficients b_n and c_n give the strength of the respective cosine and sine components of the function and d is constant. The coefficients are determined by integrating $f(t)$ over the fundamental period, T. The appropriate formulas are

$$\begin{aligned} d &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,dt, \quad b_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega_0 t\,dt, \\ c_n &= \frac{1}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega_0 t\,dt. \end{aligned} \tag{2.10}$$

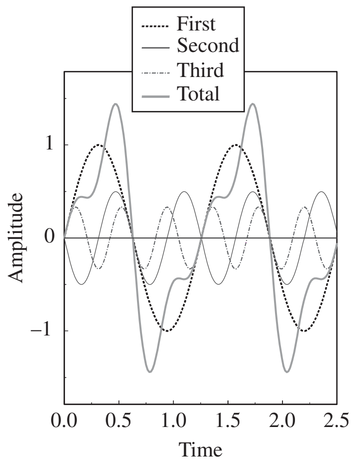

Fig. 2.9
The first three Fourier components of the sawtooth wave. The sum of these three components gives an approximation to the sawtooth shape.

These Fourier coefficients are sometimes portrayed crudely on stereo equipment as dancing bars in a dynamic bar chart that is meant to portray the strength of the music in various frequency bands.

The use of complex numbers allows Fourier series to be represented more compactly. Then Eqs. (2.9) and (2.10) become

$$f(t) = \sum_{n=-\infty}^{n=\infty} a_n e^{in\omega_0 t}, \quad \text{where} \quad a_n = \frac{\omega_0}{2\pi}\int_{-\pi/\omega_0}^{\pi/\omega_0} f(t)e^{-in\omega_0 t}dt. \tag{2.11}$$

Example 1 *Consider the time series known as the "sawtooth," $f(t) = t$ when $-\frac{T}{2} < t < \frac{T}{2}$, with the pattern repeated every period, T. Using Eq. (2.11) it can be shown that*

$$a_n = 0 \;\; \textit{for } n = 0,$$

$$a_n = \tfrac{1}{in\omega_0} \textit{ for } n = \textit{odd integer, and}$$

$$a_n = \tfrac{-1}{in\omega_0} \textit{ for } n = \textit{even integer}.$$

Through substitution and appropriate algebraic manipulation we obtain the final result:

$$f(t) = \frac{2}{\omega_0}\left[\sin \omega_o t - \frac{1}{2}\sin 2\omega_0 t + \frac{1}{3}\sin 3\omega_0 t + \cdots\right]. \tag{2.12}$$

The original function and the first three frequency components are shown in Figs. 2.9 and 2.10.

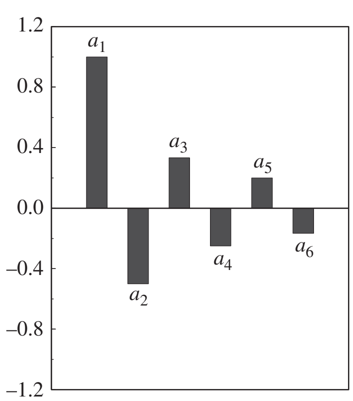

Fig. 2.10
The amplitudes of several Fourier components for the sawtooth waveform.

The time variation of the motion of the linearized version of the simple pendulum is just that of a single sine or cosine wave and therefore one frequency, the resonant frequency ω_0 is present in that motion. Obviously, the machinery of the Fourier series is unnecessary to deduce that result.

However, we now have it available as a tool for more complex periodic phenomena.

Fourier, like other contemporary French mathematicians, made his contribution to mathematics during a turbulent period of French history. He was active in politics and as a student during the "Terror" was arrested although soon released. Later when Napoleon went to Egypt, Fourier accompanied the expedition and coauthored a massive work on every possible detail of Egyptian life, *Description de l'Egypt*. This is multivolume work included nine volumes of text and twelve volumes of illustrations. During that same campaign, one of Napoleon's engineers uncovered the Rosetta Stone, so-named for being found near the Rosetta branch of the Nile river in 1899. The significance of this find was that it led to an understanding of ancient Egyptian Hieroglyphics. The stone, was inscribed with the same text in three different languages, Greek, demotic Egyptian, and Hieroglyphics. Only Greek was understood, but the size and the juxtaposition of the texts allowed for the eventual understanding of Hieroglyphics and the ability to learn much about ancient Egypt. In 1801, the victorious British, realizing the significance of the Rosetta stone, took it to the British Museum in London where it remains on display and is a popular artifact. Much later, the writings from the Rosetta stone become the basis for translating the hieroglyphics on the Rhind Papyrus and the Golenischev Papyrus; these two papyri provide much of our knowledge of early Egyptian mathematics. The French Egyptologist Jean Champollion (1790–1832) who did much of the work in the translation of Hieroglyphics is said to have actually met Fourier when the former was only 11 years old, in 1801. Fourier had returned from Egypt with some papyri and tablets which he showed to the boy. Fourier explained that no one could read them. Apparently Champollion replied that he would read them when he was older—a prediction that he later fulfilled during his brilliant career of scholarship (Burton 1999). After his Egyptian adventures, Fourier concentrated on his mathematical researches. His 1807 paper on the idea that functions could be expanded in trigonometric series was not well received by the Academy of Sciences of Paris because his presentation was not considered sufficiently rigorous and because of some professional jealousy on the part of other Academicians. But eventually Fourier was accepted as a first rate mathematician and, in later life, acted a friend and mentor to a new generation of mathematicians (Boyer and Merzbach 1991).

We have now developed the basic equations for the linearized, undamped, undriven, very simple harmonic pendulum. There are an amazing number of applications of even this simple model. Let us review some of them.

2.3 Some analogs of the linearized pendulum

2.3.1 The spring

The linearized pendulum belongs to a class of systems known as harmonic oscillators. Probably the most well known realization of a harmonic oscillator is that of a mass suspended from a spring whose restoring force is proportional to its stretch. That is

$$F_{\text{restoring}} = -kx, \tag{2.13}$$

where k is the spring constant and rate at which the spring's response increases with stretch, x. This force law was discovered by Robert Hooke in 1660. The equation of motion

$$m\frac{d^2x}{dt^2} + kx = 0 \tag{2.14}$$

is identical in form to that of the linearized pendulum and therefore its solution has corresponding properties: single frequency periodic motion, resonance, energy conservation and so forth. A schematic drawing of the spring and a graph of its force law are shown in Fig. 2.11.

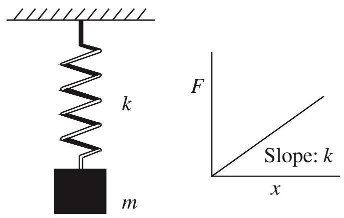

Fig. 2.11
A mass hanging from a spring. The graph shows the dependence of the extension of the spring on the force (weight). The linear relationship is known as "Hooke's law."

The functional dependence of the spring force (Eq. (2.13)) can be viewed more generally. Consider any force law that is derived from a smooth potential $V(x)$; that is $F(x) = -dV/dx$. The potential may be expanded in a power series about some arbitrary position x_0 which, for simplicity, we will take as $x_0 = 0$. Then the series becomes

$$V(x) = V(0) + V'(0)x + \frac{1}{2}V''(0)x^2 + \frac{1}{6}V'''(0)x^3 + \cdots . \tag{2.15}$$

The first term on the right side is constant and, as the reference point of a potential, is typically arbitrary and may be set equal to zero. The second, linear, term contains $V'(0)$ which is the negative of the force at the reference point. Since this reference point is, again typically, chosen to be a point of stable equilibrium where the forces are zero, this second term also vanishes. For the spring, this would be the point where the mass attached to the spring hangs when it is not in motion. Thus, the first nonvanishing term in the series is the quadratic term $\frac{1}{2}V''(0)x^2$ and comparison of it with the spring's restoring force (Eq. 2.13) leads to the identification

$$k = V''(0). \tag{2.16}$$

The spring constant is the second derivative of any smooth potential.

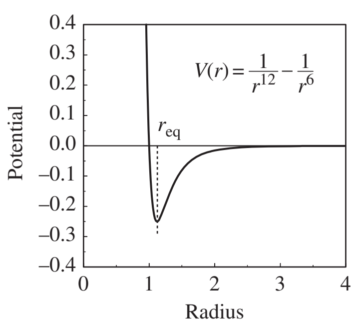

Fig. 2.12
A typical Lennard–Jones potential curve that can effectively model, for example, intermolecular interactions. For this illustration, $a = b = 1$.

Example 2 *The Lennard–Jones potential is often used to describe the electrostatic potential energy between two atoms in a molecule or between two molecules. Its functional form is displayed in Fig. 2.12 and is given by the equation*

$$V(r) = \frac{a}{r^{12}} - \frac{b}{r^6}, \tag{2.17}$$

where a and b are constants appropriate to the particular molecule. The positive term describes the repulsion of the atoms when they are too close and the negative term describes the attraction if the atoms stray too far from each other. Hence, the two terms balance at a stable equilibrium point as shown in the figure, $r_{\text{eq}} = \left(\frac{2a}{b}\right)^{1/6}$. *The second derivative of the potential may* *be evaluated at* r_{eq} *to yield the spring constant of the equivalent harmonic oscillator,*

$$V''(r_{eq}) = \frac{18b^2}{a}\left(\frac{b}{2a}\right)^{1/3} = k. \tag{2.18}$$

Knowledge of the molecular bond length provides r_{eq} *and observation of the vibrational spectrum of the molecule will yield a value for the spring constant,* k. *With just these two pieces of information, the parameters, a and b of the Lennard–Jones potential may be determined.*

The linearized pendulum is therefore equivalent to the spring in that they both are simple harmonic oscillators each with a single frequency and therefore a single spectral component. Occasionally we will refer to a pendulum's equivalent oscillator or equivalent spring, and by this terminology we will mean the linearized version of that pendulum.

2.3.2 Resonant electrical circuit

We say that a function $f(t)$ or operator $L(x)$ is linear if

$$\begin{aligned} L(x+y) &= L(x) + L(y) \\ L(\alpha x) &= \alpha L(x). \end{aligned} \tag{2.19}$$

Examples of linear operators include the derivative and the integral. But functions such as sin x or x^2 are nonlinear. Because linear models are relatively simple, physics and engineering often employ linear mathematics, usually with great effectiveness. Passive electrical circuits, consisting of resistors, capacitors, and inductors are realistically modeled with linear differential equations. A circuit with a single inductor L and capacitor C, is shown in Fig. 2.13. The sum of the voltages measured across each element of a circuit is equal to the voltage provided to a circuit from some external source. In this case, the external voltage is zero and therefore the sum of the voltages across the elements in the circuit is described by the linear differential equation

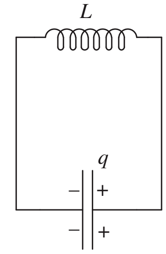

Fig. 2.13
A simple LC (inductor and capacitor) circuit.

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q = 0, \tag{2.20}$$

where q is the electrical charge on the capacitor. The form of Eq. (2.20) is exactly that of the linearized pendulum and therefore a typical solution is

$$q = q_0 \sin(\omega t + \phi), \tag{2.21}$$

where the resonant frequency depends on the circuit elements:

$$\omega = \sqrt{\frac{1}{LC}}. \tag{2.22}$$

The charge q plays a role analogous to the pendulum's angular displacement θ and the current $i = dq/dt$ in the circuit is analogous to the pendulum's angular velocity, $d\theta/dt$. All the same considerations, about the motion in phase space, resonance, and energy conservation, that previously held for the linearized pendulum, also apply for this simple electrical circuit. In a (q, i) phase plane, the point moves in an elliptical curve around the origin. The charge and current oscillate out of phase with each other. The capacitor alternately fills with positive and negative charge. The voltage across the inductor is always balanced by the voltage across the capacitor such that the total voltage across the circuit always adds to zero as expressed by Eq. (2.20). As with the spring, we will return to this electrical analog with additional complexity. For now, we turn to some applications and complexities of the linearized pendulum.

2.3.3 The pendulum and the earth

From ancient times thinkers have speculated about, theorized upon, calculated, and measured the physical properties of the earth (Bullen 1975). About 900BC, the Greek poet Homer suggested that the earth was a convex dish surrounded by the Oceanus stream. The notion that the earth was spherical seems to have made its first appearance in Greece at the time of Anaximander (610–547BC). Aristotle, the universalist thinker, quoted contemporary mathematicians in suggesting that the circumference of the earth was about 400,000 stadia—one stadium being about 600 Greek feet. Mensuration was not a precise science at the time and the unit of the stadium has been variously estimated as 178.6 meters (olympic stadium), 198.4 m (Babylonian–Persian), 186 m (Italian) or 212.6 m (Phoenician–Egyptian). Using any of these conversion factors gives an estimate that is about twice the present measurement of the earth's circumference, 4.0086×10^4 km. Later Greek thinkers somewhat refined the earlier values. Eratosthenes (276–194BC), Hipparchus (190–125BC), Posidonius (135–51BC), and Claudius Ptolemy (AD100–161) all worked on the problem. However the Ptolemaic result was too low. It is rumored that a low estimate of the distance to India, based on the Ptolemy's result, gave undue encouragement to Christopher Columbus 1500 years later.

In China the astronomer monk Yi-Hsing (AD683–727) had a large group of assistants measure the lengths of shadows cast by the sun and the altitudes of the pole star on the solstice and equinox days at thirteen different locations in China. He then calculated the length L of a degree of meridian arc (earth's circumference/360) as 351.27 li (a unit of the Tang Dynasty) which, with present day conversion, is about 132 km, an estimate that is almost 20% too high.

The pendulum clock, invented by the Dutch physicist and astronomer Christiaan Huygens (1629–1695) and presented on Christmas day, 1657, provided a powerful tool for measurement of the earth's gravitational field, shape, and density. The daily rotation of the earth was, by then, an accepted fact and Huygens, in 1673, provided a theory of centrifugal motion that required the effective gravitational field at the equator to be less than that at the poles. Furthermore, the centrifugal effect should also have the effect of fattening the earth at the equator, thereby further