

Dynamics of a Wilberforce pendulum

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The word *pendulum* is derived from the Latin word *pendulus*, which means *hanging*. One of the earliest man-made, purposeful pendulums was a seismometer designed by the Chinese scientist Zhang Heng. In 1602, Galileo discovered pendulum isochronism after gazing at a swinging lamp in the cathedral of Pisa's domed ceiling. Isochronism is an important characteristic of physical objects which states that a pendulum does not change its period, even if the amplitude changes. This behavior solely depends on the pendulum's length and the strength of gravity. In 1666, Robert Hooke studied the conical pendulum and used this device to analyze the orbital motions of planets.

The harmonic oscillator, with all its manifestations is an extensively studied system. It possesses a single degree of freedom. More complicated systems have more degrees of freedom, for example two pendulums connected by a spring [1, 2]. In such cases, the behavior of each variable influences the other which leads to a coupling between the individual degrees of freedom. Energy can toggle between the two degrees of freedom. A well known example is the Wilberforce pendulum [3], where the oscillation alternates between the longitudinal stretching of a vertical spring and the rotation of a cylindrical object hung at the end of that spring. The aim of this experiment is to investigate the dynamics of the Wilberforce pendulum.

KEYWORDS

Wilberforce Pendulum · Moment of inertia · Torque · Beats · Simple harmonic motion · Rotational motion.

APPROXIMATE PERFORMANCE TIME 8 Hours

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1 Conceptual Objectives

In this experiment, we will,

1. review the mathematical formulation for the dynamics of a Wilberforce pendulum as a typical example of two coupled oscillators,
2. understand that with two degrees of freedom, one obtains two normal modes,
3. that the Fourier transform is a useful mathematical manipulation that reveals the presence and frequencies of normal modes, and
4. learn how to compare theoretical predictions with experimental observations.

2 Mathematical background

The Wilberforce pendulum is commonly used as an important qualitative demonstration in introductory mechanics, though it has lost popularity these days! It is a nice illustration of the interaction between two coupled harmonic oscillators: the longitudinal stretching and the torsional twisting of a spiral spring attached to a mass. It also illustrates the phenomenon of beats which arises because of the intermixing of two normal modes. The pendulum is named after L. R. Wilberforce, a demonstrator in Physics at the Cavendish Laboratory in Cambridge, England, in the late 19th century [3, 4]. Wilberforce proposed the use of a loaded spiral spring to determine the Young's modulus of the spring material but also identified its potential to determine the transfer of energy between two coupled modes of a harmonic oscillator.

A spiral spring with linear spring constant, k and torsional spring constant, δ is suspended from a fixed support. A metal cylinder with mass m is attached to the free end of the spring. This is shown in Figure (1). The coordinate system is defined so that the z direction is along the axis of the spring, and the θ direction corresponds to rotation around the axis of symmetry of the system.

The equations of motion of this mass-spring system are given by, [4],

$$m\left(\frac{d^2z}{dt^2}\right) + kz + \varepsilon\theta/2 = 0, \quad (1)$$

$$I\left(\frac{d^2\theta}{dt^2}\right) + \delta\theta + \varepsilon z/2 = 0. \quad (2)$$

where $\varepsilon z\theta/2$ is the coupling between the translational and torsional motion. The parameter ε is a measure of the strength of the coupling. Remember that for an uncoupled system, the natural frequencies of the longitudinal oscillation and the torsional motion are given by,

$$\omega_z = \sqrt{\frac{k}{m}}, \quad (3)$$

$$\omega_\theta = \sqrt{\frac{\delta}{I}}. \quad (4)$$

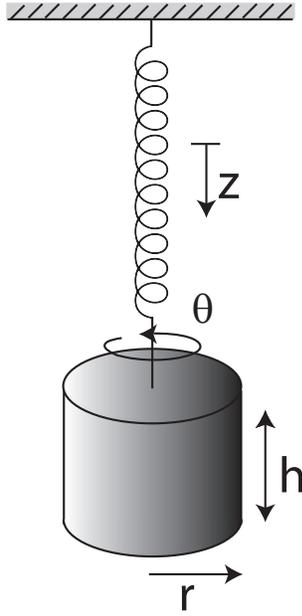


Figure 1: A Wilberforce pendulum showing oscillations along z and rotations through an angle θ . The radius and height of the cylinder are r and h respectively.

The two mutually coupled equations of motion can be simultaneously solved to determine the frequencies of the two normal modes of the system. This is what we attempt to achieve.

Q 1. Eliminate z from the coupled equations (1) and (2) obtaining the following equation in the angular variable:

$$\frac{d^4\theta}{dt^4} - (\omega_z^2 + \omega_\theta^2) \frac{d^2\theta}{dt^2} + \left(\omega_z^2 \omega_\theta^2 - \frac{\varepsilon^2}{4mI} \right) = 0. \quad (5)$$

Q 2. Now propose a solution of the form,

$$\theta(t) = Ae^{i\omega t} \quad (6)$$

and show that Eq. (5) gives rise to

$$\omega^4 - (\omega_z^2 + \omega_\theta^2) \omega^2 + \left(\omega_z^2 \omega_\theta^2 - \frac{\varepsilon^2}{4mI} \right) = 0. \quad (7)$$

This is actually a quadratic equation in ω^2 and its solution yields the frequencies of the normal modes,

$$\omega_1^2 = \frac{1}{2} \left(\omega_\theta^2 + \omega_z^2 - \left[(\omega_\theta^2 - \omega_z^2)^2 + \frac{\varepsilon^2}{mI} \right]^{1/2} \right), \quad (8)$$

$$\omega_2^2 = \frac{1}{2} \left(\omega_\theta^2 + \omega_z^2 + \left[(\omega_\theta^2 - \omega_z^2)^2 + \frac{\varepsilon^2}{mI} \right]^{1/2} \right). \quad (9)$$

Q 3. Using the frequencies of the normal modes and the definition $\omega = 2\pi f$ derive the following algebraic equation,

$$(f_1^2 - f_2^2)^2 = \left(\frac{\delta^2}{16\pi^4}\right)\left(\frac{1}{I}\right)^2 + \left(\frac{\varepsilon^2 - 2\delta k}{16\pi^4 m}\right)\left(\frac{1}{I}\right) + \left(\frac{k^2}{16\pi^4 m^2}\right). \quad (10)$$

Defining two new variables,

$$\begin{aligned} \phi &= (f_1^2 - f_2^2)^2, \\ J &= \frac{1}{I}. \end{aligned}$$

Equation (10) can be rearranged in its quadratic guise,

$$\phi = C_2 J^2 + C_1 J + C_o. \quad (11)$$

Now if we were to vary I and have some means of measuring the normal modes, f_1 and f_2 , we can fit the variation of ϕ with $J = 1/I$. The fitting constants will help us determine the parameters of the Wilberforce pendulum:

$$\delta = 4\pi^2 \sqrt{C_2}, \quad (12)$$

$$k = 4\pi^2 m \sqrt{C_o}, \quad (13)$$

$$\varepsilon = \sqrt{16\pi^4 m C_1 + 2\delta k}. \quad (14)$$

The determination of the two normal mode frequencies, f_1 and f_2 as a function of moment of inertia I , therefore, in principle, allows to measure the coupling, spring and torsional constants. This is precisely the goal of this experiment. We expect our students to observe the motion of the Wilberforce pendulum and extract the normal modes by Fourier transforming the z motion of the pendulum.

3 The Experiment

The first step is to measure the longitudinal oscillations z in real time. This could be achieved in a number of ways and your teachers would have set up the measuring devices for you. In most cases it could comprise an ultrasonic sensor that beams out ultrasound radiation which is reflected from the bottom of the pendulum mass. The echo time gives us the distance. The other option is to film the oscillating mass by placing a camera to the side of the pendulum. Then you acquire the video and if you like, use our software **Phystrack** [5] for tracing the trajectory and performing the subsequent analysis. You will have to refer to your teachers to discuss the exact technical details of the measurement process. Effectively after you've performed the experiment, you will obtain a time course of vertical positions, z .

For initiating the motion, you will disturb the equilibrium position of the spring-mass system by a twist in the horizontal plane. The system starts to oscillate and rotate, both of these kinds of motion take place in tandem.

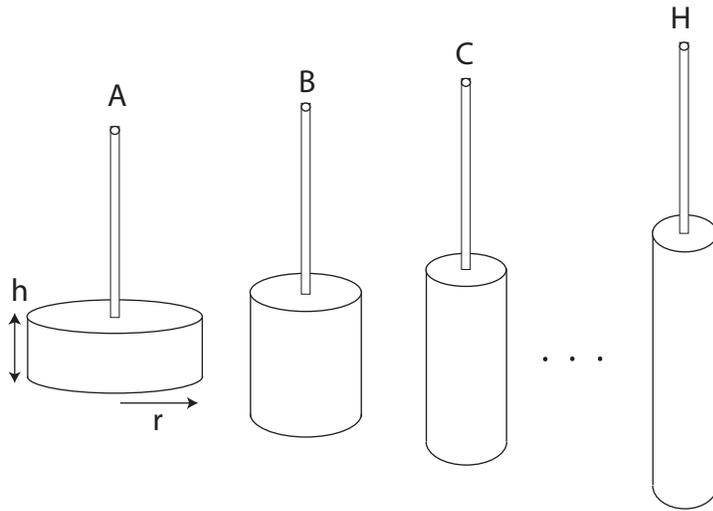


Figure 2: Set of pendulum masses. All masses m are the same within the uncertainty limits. Their moments of inertia, however, are different.

Q 4. Acquire graphs for the position z . What do you observe? By just looking at the time course of z , can you estimate the normal mode frequencies, f_1 and f_2 ? Remove the baseline of the data. This means you will adjust the offset so that the data is centred around approximately $z = 0$. Then take the Fourier transform of the data. In most cases, you should be able to easily see two peaks in the frequency spectrum which corresponds to the beating normal modes.

Fourier transformation is a cornerstone concept in experimental physics. If you are not accustomed to this concept, learn more about it from some good text on mathematical physics or signal processing. Writing a computer code to determine the Fourier transform and correctly specifying the frequency bins requires a good working knowledge of the properties of the Fourier transform and Nyquist's sampling. Learn about this for there are no shortcuts. For your convenience, given below is a piece of `Matlab` code that helps you find and plot the Fourier transform of time-domain data z , which is stored as the variable `z`. Note that `fs` is the sampling frequency which is the inverse of the time interval between which echoes or frames are acquired.

```
Nt=numel(z);
y=abs(fft(z,2^nextpow2(Nt))));
Nf=numel(y);
faxis=(0:1:Nf/2-1)*fs/Nf;
figure;
plot(faxis,y(1:Nf/2));
```

If you use our software `PhysTrack` the process is quite simple. Identify your video. Run the script `analyzeWilberforcePendulum.m` and follow the steps. After the script has run, you will need to enter the following sequence of commands to plot the time domain data and plot the Fourier transform. Note that the frame rate assumed in the commands below is 30 frames per second. The desired oscillation is along the coordinate axis y . The script for

Fourier transformation, `PhysTrack.FFT` correctly computes the frequencies of the bins and takes care of all the tedious scalings, zero-padding and truncations etc.

```
figure; plot(t,dy);  
PhysTrack.FFT(dy,30);  
figure; plot(freqAxis,amplitudes);
```

In order to experiment with various moments of inertia, I we have carefully machined cylinders with different r and h , while keeping m the same (within the limits of uncertainty). This allows us to sample various moments of inertia, obtain trajectories for them and determine how the normal mode frequencies change. Remember that for a cylinder, the moment of inertia is given by $I = mr^2/2$ and is independent of h . Our pendulums have different h to keep the masses uniform, while r determines the moment of inertia. Machining data for the pendulums, A through H, are provided in the form of a separate sheet.

Q 5. What are your observations when you experiment with varying moments of inertia? How do f_1 and f_2 change? Finally, determine the coupling, spring and torsional constants using Equation (11). Quote the uncertainties.

References

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- [5] Phystrack can odwnloaded from <https://github.com/umartechboy/PhysTrack>. You can use the script `analyzeWilberforcePendulum.m` for tracking this oscillator. A tutorial can be downloaded from <https://www.physlab.org/smart-physics/>.