Analyzing reciprocating motion: Shaft attached to a rotating wheel

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1 Abstract

In a steam engine, the expansion of compressed gasses in a cylinder is responsible for the to and fro motion of the piston. The linear motion of the piston is converted into the rotating movement of a fly-wheel by a connecting rod or crankshaft. This mechanism, nowadays, is most commonly used in internal combustion engines, in motor vehicles running on petrol or diesel. In this report we will study the reverse mechanism: Conversion of rotary motion of a wheel into the reciprocating motion of a shaft. The dependence of the motion on the lengths of various components of the setup and angle (θ) traversed by the wheel will be investigated.

2 Theory

As the wheel rotates in Figure (1b), the connecting rod pulls on or pushes the shaft horizontally, thereby changing the distance, x of the screen from the sonic detector[**1**].
Using the diagram we can write:

$$R\sin\theta \; = \; L\sin\beta \qquad \text{and} \qquad d \; = \; R\cos\theta + L\cos\beta.$$

Solving the equations simultaneously gives us the expression for d as follows:

$$\frac{d}{L} \; = \; \frac{R}{L}\cos\theta + \sqrt{1-(\frac{R}{L}\sin\theta)^2}.$$

Now,

$$d + L' + x \; = \; \mathrm{C} \qquad \Longrightarrow \qquad x \; = \; K - d.$$

Here, C is the horizontal distance between the center of the wheel and the sensor, which remains the same throughout the experiment. We have grouped it with L', another constant, in the later equation as K. Therefore we have:

$$x \; = \; -R\cos\theta - L\sqrt{1-(\frac{R}{L}\sin\theta)^2} + K. \tag{1}$$

By taking single and double time derivatives respectively we get:

$$\frac{v}{R\omega} \; = \; \frac{1}{R\omega}\frac{dx}{dt} = \sin\theta\left(1+\frac{R}{L}\cos\theta\left(1-\left(\frac{R}{L}\sin\theta\right)^2\right)^{-1/2}\right) \qquad \text{and} \tag{2}$$

$$\frac{a}{R\omega^2} \; = \; \frac{1}{R\omega^2}\frac{dv}{dt} \; = \; \frac{R}{L}\left(\sin^2\theta - \frac{\cos^2\theta}{1-\frac{R}{L}\sin^2\theta}\right)\left(1-\left(\frac{R}{L}\sin\theta\right)^2\right)^{-1/2} - \cos\theta, \tag{3}$$

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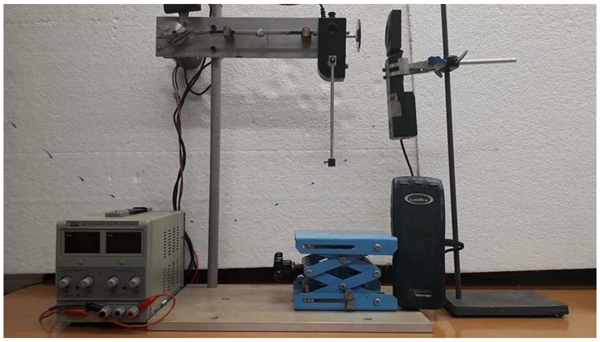
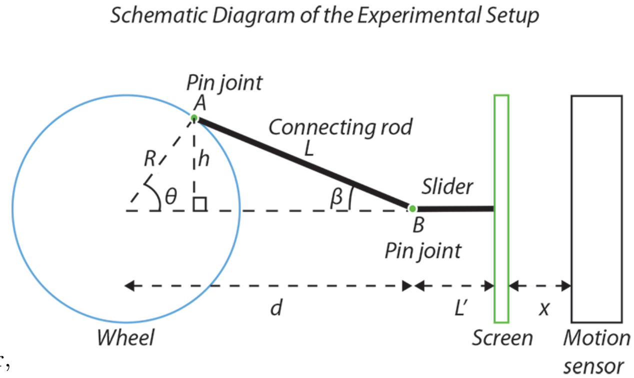


(a) The pendulum setup used in this experiment. However, our experiment only involves the to and fro motion of the black disc (screen) attached to the right end of the shaft.



(b)

Figure 1

Voltage (v)	Angular Frequency, ω (rad/s)	Period, T (s)
30	8.266 ± 0.002	0.760 ± 0.001
18	4.744 ± 0.002	1.324 ± 0.004

Table 1: Values of the angular frequency and their corresponding time periods obtained from our curve fittings

where, v is the velocity of the shaft, a is the acceleration, and ω is the angular frequency of the wheel, which is constant and depends on the electrical power supplied.

3 The Experiment

3.1 Motion sensor

The Vernier Motion Detector used, is an ultra-sonic device that determines the position of the object in front of it by the echo time of ultra-sonic waves. It has a resolution of 1 mm.

3.2 Procedure

Figure (1a) shows the complete setup of the experiment. The wheel is powered by a DC power supply (HY5003). The Vernier Motion Detector was placed with its center coinciding with that of the black screen for maximum echo. It was interfaced with the computer through the Vernier LabPro Interface.

Before taking readings for a particular value of voltage, the sensor's distance from the screen had to be adjusted for an accurate detection of motion and then maintained for the rest of the experiment. This is necessary because if the distance is too short then the time between successive echos is greatly diminished and the detector is unable to differentiate between incoming signals. In the opposite case, the resolution of the detector is diminished and minor changes in the disc motion go undetected. This required observing the signal for various distances before deciding upon the most suitable one. In our case, suitable distances were found to be around 27 cm and $25-26$ cm for 30 V and 18 V respectively.

The lengths, L and R were measured using vernier calipers to be:

$$R \;=\; (15.00 \pm\; 0.5)\text{ mm} \quad\text{and}\quad L \;=\; (36.00 \pm\; 0.5)\text{ mm}. \tag{4}$$

3.3 Obtaining Results

The signals were recorded on Logger Pro software and plots for the position, velocity and acceleration of the shaft, w.r.t time were obtained at 30 V and 18 V electric supply. Table (1) lists the results. The equations, derived in the first section, were used along with Eq. (4) for this purpose. and are stated as follows.

Angular Frequency ω (rad/s)	x_{max} (cm)	x_{min} (cm)	$\Delta x(cm)$	A
4.744 ± 0.002	25.5	22.8	2.70	0.900
8.266 ± 0.002	26.6	24.2	2.40	0.900

Table 2: The maximum and minimum displacement of the screen from the sensor. Δx should ideally be equals to $2R$.

Angular Frequency, ω (rad/s)	v_{max} (cm/s)
4.744 ± 0.002	6.70
8.266 ± 0.002	10.0

Table 3: The maximum magnitudes of velocity obtained from the best-fit curve.

3.3.1 Position

The best-fit curve equation plotted w.r.t time in Figure (3a) and Figure (4a) is:

$$x \ = \ A\left(- R\cos t\omega - L\sqrt{1 - \left(\frac{R}{L}\sin t\omega\right)^2}\ \right) + K \tag{5}$$

The fitting parameters, ω, A (unitless), and K (m) were adjusted manually to fit our experimental results in both the cases. If we compare Eq. (5) to Eq. (1), A comes out to be unity and K represents a constant distance. Also,

$$\theta = t \times \omega, \tag{6}$$

where, t is the time elapsed in seconds, after the wheel starts rotating, and ω is the angular frequency in $\mathrm{rad\,s^{-1}}$.

The maximum distance of the screen from the sensor occurs at $\theta = \pi$ rad and the minimum distance at 0 rad. Applying this to Eq. (5) and using the values of fitting parameters, we get the following equations. The values are given in Table (2).

$$x_{min} = A(-R - L) + K \quad \text{and} \quad x_{max} = A(R - L) + K$$

and

$$\Delta\ x \ = \ A(2R).$$

3.3.2 Velocity

The best fit equation for velocity plotted in Figure (3b) and Figure (4b):

$$v \ = \ B\sin t\omega\left(1 + \frac{R}{L}\cos t\omega\left(1 - \left(\frac{R}{L}\sin t\omega\right)^2\right)^{-1/2}\right) \tag{7}$$

The equation uses the same value of ω, as Eq. (5). The fitting parameter, B, was adjusted manually.

3.3.3 Acceleration

The best-fit curve equation used in Figure (3c) and Figure (4c):

$$a = D\left(\frac{R}{L}\left(\sin^2 t\omega - \frac{\cos^2 t\omega}{1 - \frac{R}{L}\sin^2 t\omega}\right)\left(1 - \left(\frac{R}{L}\sin t\omega\right)^2\right)^{-1/2} - \cos t\omega\right) \tag{8}$$

Units of D are: $\mathrm{cm/s^2}$ and the value of ω same as deduced in Eq. (5).

Coordinate axis	Position at $\pi/2$ rad	Mid-point displacement	Exp. value
x	1.57	1.36	1.46
y	0.245	0.242	0.244

Table 4: The intersection points for half cycle from θ in the range 0 to π rad.

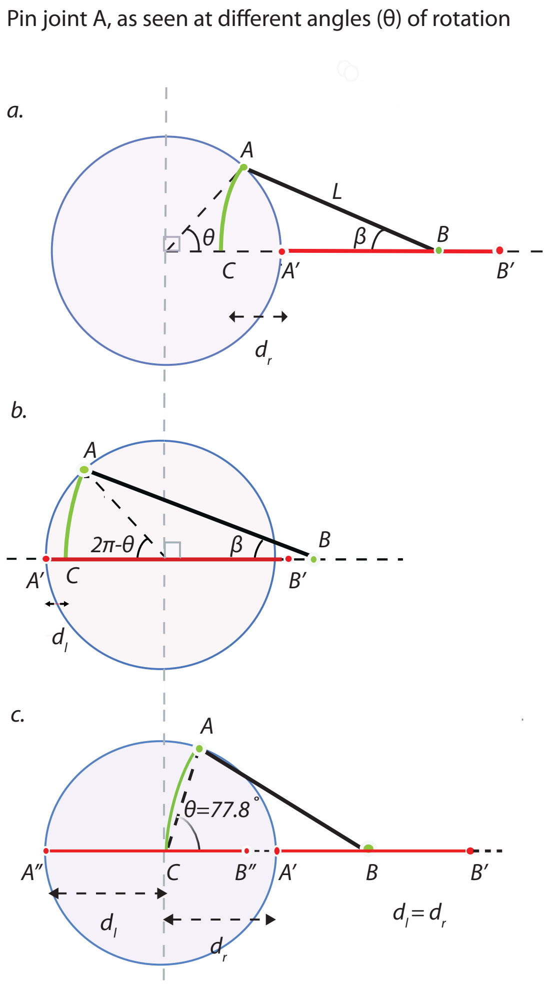
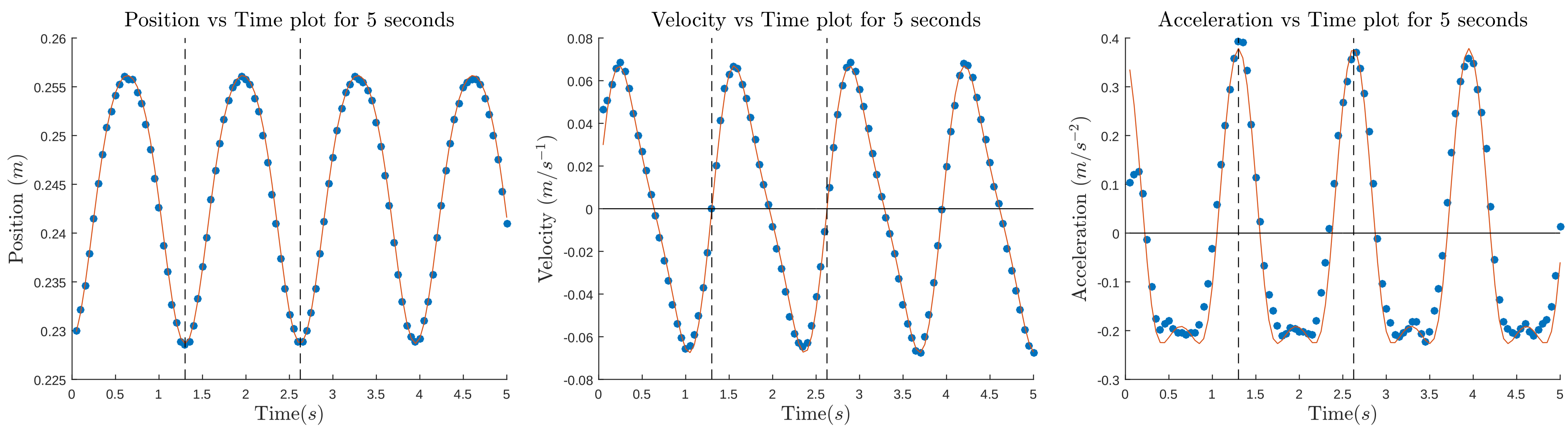


Figure 2: Figure (a) and Figure (b) shows the Pin joints A at equal height above the horizontal ($\theta = 2\pi - \theta$) in the right and left semicircle of the wheel respectively. If the connecting rod at AB, were to swing around a fixed point B towards the horizontal axis, it will cut the axis at C. $A'B'$ in Figure (a) and Figure (b), are the positions of the connecting rod corresponding to the minimum and maximum displacement of the shaft from the sensor respectively. d_r and d_l give the linear displacement of the shaft when the rotation of the wheel brings A to A'. Figure (c) gives the angular displacement of A corresponding half the linear displacement of the shaft between 0 rad to π rad. It can be seen that this doesn't occur at $\pi/2$ rad as one would have expected, but occurs at an acute angle. The linear displacement of the shaft will be equal, if A were to move either towards A'' or A' because C lies on exactly the center of the wheel.

4 Discussion

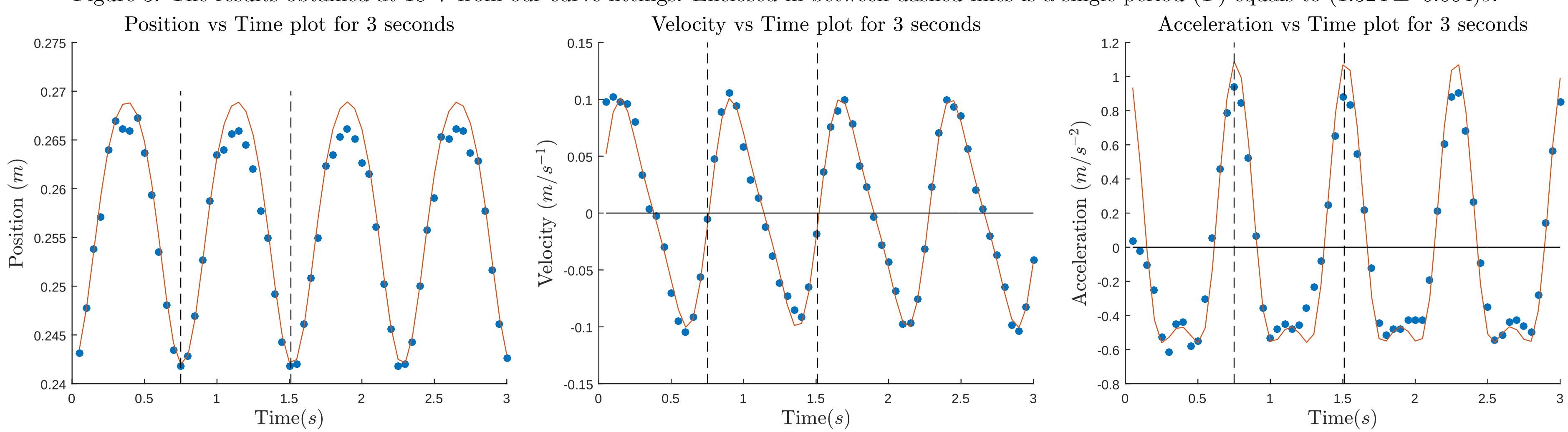
As Pin-joint A traverses one full rotation around the wheel, the distance it covers in the left semicircle of rotation is less than the distance it covers in the right semicircle. This is evident in in Figure (5). Figure (2) gives an intuitive understanding of why this is so. We will analyze the behaviour of the shaft rotating at $\omega = (4.744 \pm\ 0.002)s$ over one complete period. We can see in Figure (2) that d_l will always be less than d_r meaning that although A spends equal amount of time in both halves (ω is constant), the to and fro displacement the shaft covers as A spans the right side is greater than when it is moving across the left. Therefore the value of θ corresponding to the expectation value of position is an acute angle, as stated in Table (4).
Finding the expectation value of Pin-joint A using Figure (5) :

$$\text{Exp. value of Position} = \frac{\text{Area under the plot}}{2\pi} = 0.2438 \text{ rad}$$



(a) (b) (c)

Figure 3: The results obtained at 18 V from our curve fittings. Enclosed in between dashed lines is a single period (T) equals to $(1.324 \pm\ 0.004)s$.



(a) (b) (c)

Figure 4: The results obtained at 30 V. In this case, a single period of the rotation is for a time period, $T = (0.760 \pm\ 0.001)s$.

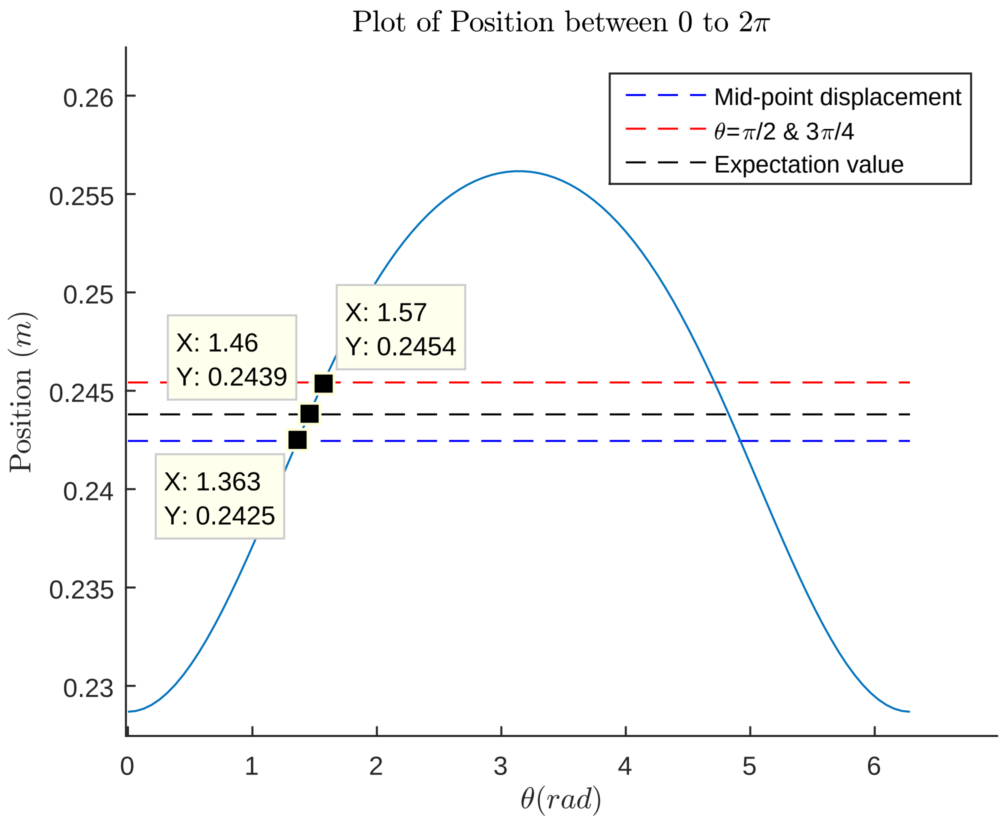
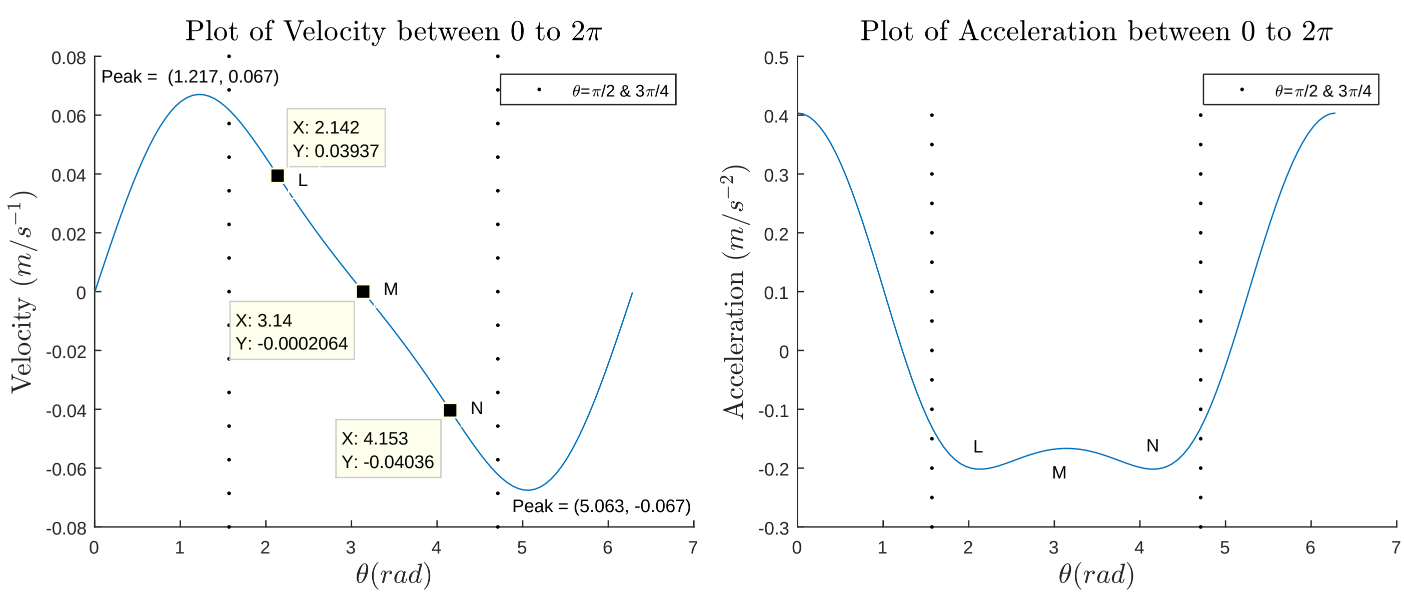


Figure 5: A comparison of the expectation value, the mid-point of displacement between $\theta = 0$ rad and π rad and the position at $\pi/2$ rad. The plot is symmetric about $\theta = \pi$ and the remaining intersection points can be easily calculated.



(a)

(b)

Figure 6: As θ increases from 0 rad, the velocity of the shaft increases in the positive direction (decreasing acceleration) until in reaches it's maximum value at 1.217 rad (69.70°) (acceleration =0 rad/s^2). After this, the shaft starts to slow down (deceleration) while still moving in the positive direction. The deceleration is maximum at point L, after which it becomes increasingly lower till M is reached. After this, the shaft accelerates in the negative direction with increasing magnitude till N. From here the acceleration reduces to zero at 5.063 rad (290.2°). This is the maximum velocity in the negative direction). Beyond this point, the shaft decelerates till the velocity is 0 rad/s, before the start of the next cycle.

4.1 Understanding the extreme values of velocity

It can be seen in Figure (6a) that the extreme values of the velocity occur in the right semi circle of rotation. In Figure (1b) if $\theta = \theta'$ is such that the connecting rod is tangential to the wheel, then the centripetal velocity ($R \times \omega$) will be the same as the velocity of the rod being pulled/pushed at a corresponding angle β'. In that case, the speed of the shaft in the forward/backward direction is simply:

$$|v| = R\omega \cos\beta \text{ where } \beta = \pi/2 - \theta'. \tag{9}$$

We can say that at θ', the rotary motion of the wheel is being converted most effectively into the linear motion of the shaft and the velocity of the shaft in either direction is maximum.
We use values of θ' from Figure (6a) and inserting them into Eq. (9). We then have:

$$|v| = R\omega \cos(0.353) = 0.668 \text{ for } \theta = 1.217 \text{ rad} \tag{10}$$
$$|v| = R\omega \cos(-0.349) = 0.668 \text{ for } \theta = 5.063 \text{ rad} \tag{11}$$

Our answers match the values in Table (3) with just an error of 0.3%.

4.2 Dependence of motion on length segments

It can be shown that the disparity in the maximum displacement of the shaft toward the left and right depends upon the ratio $\alpha = R/L$. Using Eq. (1) we calculate the ratio of the maximum displacement to the left, d_l to the total peak to peak displacement, $2R$, of the shaft:

$$\frac{d_l}{2R} = \frac{x(\theta = \pi) - x(\theta = \pi/2)}{2R} = \left(\frac{\alpha - 1 + \sqrt{1-\alpha^2}}{\alpha}\right)\frac{1}{2R},$$

The relation is mapped in Figure (7). For our setup:

$$\alpha = 0.405 \pm 0.020 \text{ and}$$
$$\frac{d_l}{2R} = 0.394 \pm 0.005$$

The ratio R/L must be greater than unity, otherwise as the Pin joint A moves from 0 rad to $\pi/2$ rad the shaft will be pulled to the left. Once A has crossed $\pi/2$ rad, the connecting rod will simply swing about point B in all consecutive rotations, concentric to the wheel and no reciprocating motion occurs (consult Figure(1b). At very large value of L, the displacement on both sides of the wheel is the same where as for $L = R$ it is zero.

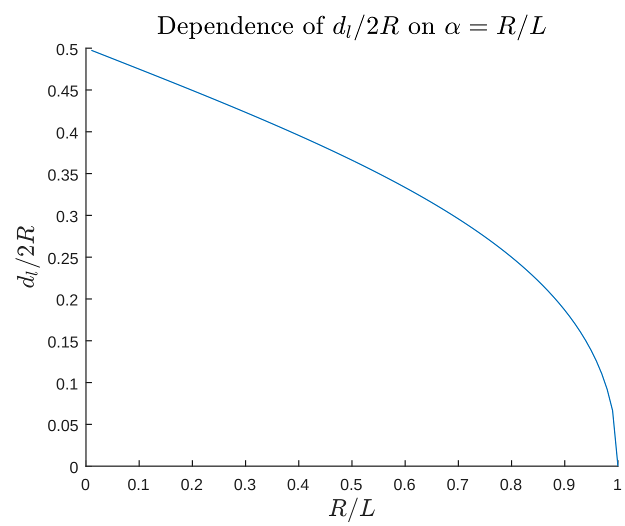


Figure 7

5 Conclusion

We investigated the motion of a shaft undergoing reciprocating motion and concluded that the motion is not symmetric about the vertical axis through the center of the wheel. This is proven by the fact that the expectation value of position of the shaft is such that Pin joint A occurs at an acute angle. The ratio, R/L, was found to be crucial to this phenomenon and for connecting rods having lengths much greater than the radius of the wheel ($L >> R$), the maximum displacement on either side of the vertical was found to be the same.