Assignment 1: Quantum States
Due Date: September 30, 2019

1. The polarization of a photon is a two-level quantum system. The plane of polarization is specified as a quantum state

$$|\psi\rangle = C_H\,|H\rangle + C_V\,|V\rangle,$$

where $|H\rangle$ and $|V\rangle$ correspond to one possible basis and represent horizontal and vertical polarizations, i.e., $|H\rangle = |\theta = 0\rangle$ and $|V\rangle = |\theta = \pi/2\rangle$. This system is mathematically identical (isomorphic) to a spin-1/2 system studied in class. You are invited to read about polarization from the manual uploaded on https://www.physlab.org/experiment/light-is-a-transverse-wave/.

Suppose $|\theta\rangle$ represents the state of a beam of photons linearly polarized at an angle of θ from the horizontal.

(a) Write $|\theta\rangle$ as a linear combination of $|H\rangle$ and $|V\rangle$.

(b) What is the probability that a photon in the state $|\theta\rangle$ will be measured to have vertical polarization? The equivalent of a Stern-Gerlach apparatus is a polarization analyzer PA_θ that can project the input state into either $|\theta\rangle$ or its orthogonal state $|\theta + 90°\rangle$.

(c) What is the probability that a photon in the state $|\theta\rangle$ will be measured to have linear polarization along +45°? (d) What is the probability that a photon in the state $|\theta\rangle$ will be measured to have right-circular polarization, which is given by $\frac{1}{\sqrt{2}}(|H\rangle - i|V\rangle)$.

(e) If a beam of photons in state $|V\rangle$ is sent through a series of two polarization beam splitters (polarization analyzers), as illustrated in Fig. 1, then

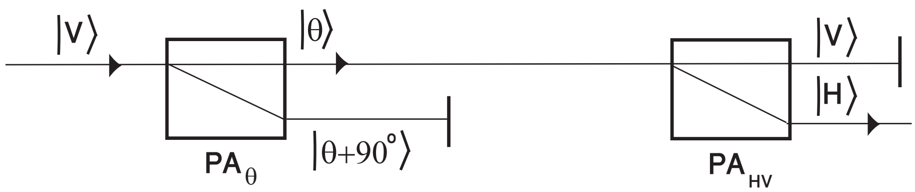

Fig. (1)

i. What fraction of the input photons will survive to the final output?

ii. At what angle θ must the PA_θ be oriented so as to maximize the number of photons that are transmitted by the PA_{HV}?

iii. What fraction of the photons is transmitted if the PA_θ is simply removed from the experiment?

2. Consider the following unnormalized state vectors of a 2-dimensional quantum system:

$$\begin{aligned} |\psi_1\rangle &= 3\,|+\rangle + 4\,|-\rangle \\ |\psi_2\rangle &= |+\rangle + 2i\,|-\rangle \\ |\psi_3\rangle &= 3\,|+\rangle - e^{i\pi/3}\,|-\rangle \end{aligned}$$

(a) Normalize each state vector.

(b) For each state vector, calculate the probability that the spin component is up or down along each of the three Cartesian axes. Use bra-ket notation for the entire calculation.

(c) Write each normalized state in matrix (vector) notation. To do so, you first need to represent the basis states.

3. Consider the following three quantum states:

$$\begin{aligned} |\psi_1\rangle &= \frac{1}{\sqrt{3}}\,|+\rangle + i\frac{\sqrt{2}}{\sqrt{3}}\,|-\rangle \\ |\psi_2\rangle &= \frac{1}{\sqrt{5}}\,|+\rangle - i\frac{2}{\sqrt{5}}\,|-\rangle \\ |\psi_2\rangle &= \frac{1}{\sqrt{2}}\,|+\rangle + \frac{e^{i\pi/4}}{\sqrt{2}}\,|-\rangle \end{aligned}$$

(a) For each of the $|\psi_i\rangle$ above, find the normalized vector $|\phi_i\rangle$ that is orthogonal to it.

(b) Calculate the inner product $\langle\psi_i|\psi_j\rangle$ for i and $j = 1, 2, 3$.

4. Consider a quantum system with an observable A that has three possible measurement results: a_1, a_2, and a_3. Normalize where necessary.

(a) Write down the three kets $|a_1\rangle$, $|a_2\rangle$, and $|a_3\rangle$ corresponding to these possible results using matrix notation.

(b) The system is prepared in the state

$$|\psi\rangle = 1\,|a_1\rangle - 2\,|a_2\rangle + 5\,|a_3\rangle.$$

Write this state in matrix notation, and calculate the probabilities of all possible measurement results of the observable A. Plot a histogram of the predicted measurement results.

(c) In a different experiment, the system is prepared in the state

$$|\psi\rangle = 2\,|a_1\rangle - 3i\,|a_2\rangle.$$

Write this state in matrix notation, and calculate the probabilities of all possible measurement results of the observable A. Plot a histogram of the predicted measurement results.

5. Consider a quantum system described by the basis $|a_1\rangle$, $|a_2\rangle$, and $|a_3\rangle$. The system is initially in a state

$$|\psi_i\rangle = \frac{1}{\sqrt{3}}\,|a_1\rangle + \sqrt{\frac{2}{3}}\,|a_2\rangle.$$

Find the probability that the system is measured to be in the final state

$$|\psi\rangle = \frac{1+i}{\sqrt{3}}\,|a_1\rangle + \sqrt{\frac{1}{6}}\,|a_2\rangle + \sqrt{\frac{1}{6}}\,|a_3\rangle.$$

6. The spin components of a beam of atoms prepared in the state $|\psi_{in}\rangle$ are measured, and the following experimental probabilities are obtained:

$$p_+ = \frac{1}{2} \qquad p_{+x} = \frac{3}{4} \qquad p_{+y} = 0.067$$
$$p_- = \frac{1}{2} \qquad p_{-x} = \frac{1}{4} \qquad p_{-y} = 0.033$$

From the experimental data, estimate the input state, that is, write your best guess of $|\psi_{in}\rangle$ represented in a suitable basis.