Assignment 2: Ride with Matrix and Dirac Notations
Due Date: October 7, 2019

1. **Kets Are Old Friends**

   As queer as they seem, kets play by the same rules as good old column vectors. In fact, represented via matrix notation, kets *are* column vectors. Consider the following:

$$|\psi\rangle \doteq \begin{bmatrix} -3i \\ 2+i \\ 4 \end{bmatrix},$$

   and

$$|\phi\rangle \doteq \begin{bmatrix} 2 \\ -i \\ 2-3i \end{bmatrix},$$

   where "$\doteq$" stands for "is represented by."

   (a) Represent $\langle\phi|$ in matrix notation. Remember to use "$\doteq$."

   (b) Calculate inner product $\langle\phi|\psi\rangle$.

   (c) In view of matrix notation, why would $|\psi\rangle|\phi\rangle$ and $\langle\psi|\langle\phi|$ not make sense?

2. **The Reunion Continues**

   Spending some more time with kets would not go in vain. Consider the states

$$|\psi\rangle = 3i\,|\phi_1\rangle - 7i\,|\phi_2\rangle$$

   and

$$|\chi\rangle = -\,|\phi_1\rangle + 2i\,|\phi_2\rangle,$$

   where $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthonormal.

   (a) Taking $\{|\phi_1\rangle, |\phi_2\rangle\}$ as the basis, use matrix notation to write $|\psi\rangle$ and $|\chi\rangle$ as column vectors. Again, do not forget to use "$\doteq$."

   (b) Calculate inner products $\langle\psi|\chi\rangle$ and $\langle\chi|\psi\rangle$.

   (c) How are they related?

3. **Enter the Outer Product**

   Like the inner product, the outer product is an operation involving a bra and a ket.

However, whereas the former yields a complex number, the latter gives a matrix (given that matrix notation is used). Consider the following states:

$$|\psi\rangle = i\,|\phi_1\rangle + 3i\,|\phi_2\rangle - |\phi_3\rangle,$$

and

$$|\chi\rangle = |\phi_1\rangle - i\,|\phi_2\rangle + 5i\,|\phi_3\rangle,$$

where $|\phi_1\rangle$, $|\phi_2\rangle$, and $|\phi_3\rangle$ are orthogonal and normalized.

(a) Calculate $|\psi\rangle\langle\chi|$ and $|\chi\rangle\langle\psi|$. Are they equal?

(b) Find the transpose of $|\psi\rangle\langle\chi|$, and complex conjugate every entry. How does your result relate to $|\chi\rangle\langle\psi|$.

Note: Use matrix notation throughout.

4. **Yet Another Drill**

Consider the states

$$|\psi\rangle \doteq \begin{bmatrix} 5i \\ 2 \\ -i \end{bmatrix}$$

and

$$|\phi\rangle \doteq \begin{bmatrix} 3 \\ 8i \\ -9i \end{bmatrix}.$$

(a) Find $|\psi\rangle^*$ and $\langle\psi|$.

(b) Is $|\psi\rangle$ normalized? If not, normalize it.

(c) Are $|\psi\rangle$ and $|\phi\rangle$ orthogonal?

5. **Gauging Your Attention**

Let $|1\rangle$, $|2\rangle$, $|3\rangle$, ... , and $|N\rangle$ form an orthonormal basis for a Hilbert space. Then, an arbitrary vector, $|\psi\rangle$, in that space can be written down as a linear combination of these kets, that is,

$$|\psi\rangle = c_1\,|1\rangle + c_2\,|2\rangle + c_3\,|3\rangle + \ldots + c_N\,|N\rangle,$$

where $c_1, \ldots, c_N \in \mathbb{C}$. In the class, I hinted at a proof for the fact that

$$c_i = \langle i|\psi\rangle,$$

where $i = 1, \ldots, N$. Harnessing the power of the Dirac notation, produce a detailed proof.