Assignment 3: Operators Gala
Due Date: October 23, 2019

1. **The Omnipresent (15 Points)**

   Unitary operators comprise an important class of quantum operators. From basis transformations to rotations on the Bloch sphere to time evolution of quantum states (which has not been discussed as yet), unitary operators come right on the spot. Despite the impressive gamut of its applications, a unitary operator has a simple, down-to-earth definition: it is an operator whose adjoint is its inverse. To be precise, if $\hat{U}$ is a unitary operator, then

   $$\hat{U}^\dagger \hat{U} = \hat{\mathbb{1}} = \hat{U}\hat{U}^\dagger.$$

   In this problem, you would prove some important properties of unitary operators.

   (a) **(5 Points)** Prove that every eigenvalue of a unitary operator, $\hat{U}$, is just a phase. That is, it has the form $e^{i\phi}$.

   (b) **(5 Points)** Show that the product of two unitary operators, $\hat{U}_1$ and $\hat{U}_2$, is also unitary.

   (c) **(5 Points)** Show that the action of a unitary operator, $\hat{U}$, does not change the norm of a state.

2. **Daggers Won't Work (15 Points)**

   Like unitary operators, Hermitian operators too constitute an important class of quantum operators. They connect the mathematical structure of quantum mechanics to measurements in the real world by representing what we call "observables." Mathematically, an operator, $\hat{A}$, is said to be Hermitian if

   $$\hat{A}^\dagger = \hat{A}.$$

   That is, the adjoint of a Hermitian operator equals the operator itself ("Daggers Won't Work"). In this problem, you would prove some important properties of Hermitian operators.

   (a) **(5 Points)** Prove that every eigenvalue of a Hermitian operator, $\hat{A}$, is real.

   (b) **(5 Points)** Show that the product of two Hermitian operators, $\hat{A}_1$ and $\hat{A}_2$, is not

Hermitian.

(c) **(5 Points)** Show that for a Hermitian operator, $\hat{A}$, the eigenvectors corresponding to different eigenvalues are orthogonal to each other.

3. **Unitary Operators in Action (15 Points)**

Now, we proceed to see the role a unitary operator performs in a basis transformation. Let $\{|1\rangle, |2\rangle, |3\rangle, \ldots, |N\rangle\}$ and $\{|\tilde{1}\rangle, |\tilde{2}\rangle, |\tilde{3}\rangle, \ldots, |\tilde{N}\rangle\}$ be two bases for a Hilbert space.

(a) **(5 Points)** Show that the similarity matrix, $\hat{S}$, for transformation from the former to the latter basis is unitary.
Hint: Use the elegance of the Dirac notation to your advantage.

A polarization rotator is an optical instrument that rotates the plane of polarization of a photon through an angle.

(b) **(5 Points)** If the angle is α (counterclockwise) for a particular rotator, write the operator representing that rotator in the basis $\{|H\rangle, |V\rangle\}$. You may use matrix notation.

(c) **(5 Points)** An old, crazy scientist comes along and insists that you use a basis that is rotated through an angle of $\frac{\pi}{4}$ with respect to the basis $\{|H\rangle, |V\rangle\}$. Respecting his old age, represent the operator you have just written in the basis of his choice.
Note: I want you to use the similarity matrix.

4. **Hermitian Gymnastics (25 Points)**

When it reads "Gymnastics," it means genuinely so.

(a) **(3 Points)** Discuss the Hermicity of the operators $\hat{A}+\hat{A}^\dagger$, $i(\hat{A}+\hat{A}^\dagger)$, and $i(\hat{A}-\hat{A}^\dagger)$.

Consider a two-dimensional Hilbert space where a Hermitian operator, $\hat{A}$, is defined in the following manner:

$$\hat{A}|\phi_1\rangle = |\phi_1\rangle,$$

and

$$\hat{A}|\phi_2\rangle = -|\phi_2\rangle,$$

where $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthonormal.

(b) **(2 Points)** Do $|\phi_1\rangle$ and $|\phi_2\rangle$ form a basis?

Hint: Spectral Decomposition

(c) **(5 Points)** Consider operator $\hat{B} = |\phi_1\rangle\langle\phi_2|$. Is $\hat{B}$ Hermitian? Show that $\hat{B}^2 = \hat{0}$.

(d) **(2 Points)** Calculate $\hat{B}\hat{B}^\dagger$ and $\hat{B}^\dagger\hat{B}$.

(e) **(3 Points)** Is $\hat{B}\hat{B}^\dagger + \hat{B}^\dagger\hat{B}$ Hermitian?

(f) **(3 Points)** Is $\hat{B}\hat{B}^\dagger - \hat{B}^\dagger\hat{B}$ Hermitian? If not, what kind of operator is it?

(e) **(2 Points)** Consider operator $\hat{C} = \hat{B}\hat{B}^\dagger + \hat{B}^\dagger\hat{B}$. Show that $\hat{C}|\phi_1\rangle = |\phi_1\rangle$ and $\hat{C}|\phi_2\rangle = |\phi_2\rangle$.

Consider the operator

$$\hat{A} = |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| + |\phi_3\rangle\langle\phi_3| - i|\phi_1\rangle\langle\phi_2| - |\phi_1\rangle\langle\phi_3| + i|\phi_2\rangle\langle\phi_1| - |\phi_3\rangle\langle\phi_1|,$$

where $|\phi_1\rangle$, $|\phi_2\rangle$, and $|\phi_3\rangle$ form an orthonormal basis for the underlying Hilbert space.

(a) **(3 Points)** Is $\hat{A}$ Hermitian? Calculate $\hat{A}^2$. Is it a projection operator?

(b) **(2 Points)** Find a 3×3 matrix representing $\hat{A}$ in the basis $\{|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle\}$.

5. **Gram-Schmidt Process: A Primer (10 Points)**

"Gram-Schmidt process" is the fancy name of an algorithm that is used to orthonormalize a linearly independent set of kets. More precisely, if set $\{|1\rangle, |2\rangle, |3\rangle, \dots, |P\rangle\}$ is linearly independent but not orthonormal, then the Gram-Schmidt process can create an orthonormal set, $\{|\tilde{1}\rangle, |\tilde{2}\rangle, |\tilde{3}\rangle, \dots, |\tilde{P}\rangle\}$, whose span equals that of $\{|1\rangle, |2\rangle, |3\rangle, \dots, |P\rangle\}$. The algorithm is given below:

$$\begin{aligned}
|\tilde{1}\rangle &= |1\rangle, \\
|\tilde{2}\rangle &= |2\rangle - \frac{\langle\tilde{1}|2\rangle}{\langle\tilde{1}|\tilde{1}\rangle}|\tilde{1}\rangle, \\
|\tilde{3}\rangle &= |3\rangle - \frac{\langle\tilde{1}|3\rangle}{\langle\tilde{1}|\tilde{1}\rangle}|\tilde{1}\rangle - \frac{\langle\tilde{2}|3\rangle}{\langle\tilde{2}|\tilde{2}\rangle}|\tilde{2}\rangle, \\
&\vdots \\
|\tilde{P}\rangle &= |P\rangle - \frac{\langle\tilde{1}|P\rangle}{\langle\tilde{1}|\tilde{1}\rangle}|\tilde{1}\rangle - \frac{\langle\tilde{2}|P\rangle}{\langle\tilde{2}|\tilde{2}\rangle}|\tilde{2}\rangle - \dots - \frac{\langle\widetilde{P-1}|P\rangle}{\langle\widetilde{P-1}|\widetilde{P-1}\rangle}|\widetilde{P-1}\rangle.
\end{aligned}$$

You are given the following linearly independent set:

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ i \end{pmatrix}, \begin{pmatrix} -1 \\ i \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1+i \end{pmatrix} \right\}$$

The three vectors in the set are matrix representations of three kets in a particular basis. Notice that they are neither orthogonal nor normal. Use the Gram-Schmidt process to orthonormalize them.

Note: Make sure that every vector in the final set of vectors you obtain is normalized.