$\underline{Q_1}$ PHY300 SOLUTION: HOMEWORK 1

i) mean $= \sum_{i=1}^{N} \frac{x_i}{N} \Rightarrow 50.74\ \mu C$

ii) standard deviation $= \sigma = \sqrt{\frac{(x_i-\bar{x})^2}{N-1}} = 5.78\ \mu C$

iii) for $62.1 \times 10^{-6} C$,

$$Z = \frac{62.1 - \bar{x}}{\sigma} = 1.96.$$

From chauvenet's criterion table, we know that for a data set of 7 values the allowed z value should be less than 1.803. Since $1.96 > 1.803$, the value is an outlier and thus, should be ignored.

iv) The new value for mean is, $\mu = 48.8\ \mu C$.

v) The new value for standard dev, $\sigma = 3.16\ \mu C$.

$\underline{Q_2}$

$$P_U(x; \bar{x}, a) = \begin{cases} 1/a & \text{if } \bar{x}-a/2 \leq x \leq \bar{x}+a/2 \\ 0 & \text{otherwise} \end{cases}$$

(i) $\int_{-\infty}^{\infty} P_u(x;\bar{x},a)\, dx$

$$\int_{\bar{x}-a/2}^{\bar{x}+a/2} \frac{dx}{a} = \frac{1}{a}\left[\left(\bar{x}+\frac{a}{2}\right)-\left(\bar{x}-\frac{a}{2}\right)\right] = 1 \quad \text{normalized.}$$

ii) $\mu = \int_{-\infty}^{\infty} x P_u(x;\bar{x},a)\, dx \Rightarrow \int_{\bar{x}-a/2}^{\bar{x}+a/2} \frac{x}{a}\, dx$

$$= \frac{1}{2}\frac{x^2}{a}\Bigg|_{\bar{x}-a/2}^{\bar{x}+a/2} = \bar{x} \quad \text{Q.E.D.}$$

(iii) Standard deviation,

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\bar{x})^2 P_u(x;\bar{x},a)\,dx$$

$$= \int_{\bar{x}-\frac{a}{2}}^{\bar{x}+a/2} (x-\bar{x})^2 \cdot \frac{1}{a}\,dx$$

$$= \frac{1}{a}\int_{\bar{x}-\frac{a}{2}}^{\bar{x}+\frac{a}{2}} (x^2 - 2x\bar{x} + \bar{x}^2)\,dx$$

$$\sigma^2 = \frac{a^2}{12}$$

$$\sigma = \frac{a}{\sqrt{12}} \quad \text{Q.E.D.}$$

$\underline{Q_3}$. $\delta^2 = E[(\bar{x} - x_T)^2]$

$= E(\bar{x}^2 - 2\bar{x}x_T + x_T^2)$

$= E\left[\left(\sum_i \frac{x_i}{N}\right)^2 - 2\left(\sum_i \frac{x_i}{N}\right)x_T + x_T^2\right]$

$= E\left[\frac{\sum_i x_i^2}{N^2} - \sum_{j \neq k} \frac{x_j x_k}{N^2} - 2\sum_i \frac{x_i x_T}{N} + x_T^2\right]$

$= \frac{1}{N^2} E\left(\sum_i x_i^2\right) + \frac{1}{N^2} \sum_{j \neq k} E(x_j x_k) - \frac{2}{N}\sum_i E(x_i x_T) + x_T^2$

$= \frac{1}{N^2} \sum_i \left(E[x_i^2]\right) + \frac{1}{N^2} \sum_{j \neq k} E(x_j x_k) - \frac{2}{N}\sum_i E(x_i) x_T + x_T^2$

$= \frac{1}{N^2} N E(x_i^2) + \frac{1}{N^2} N(N-1) E(x_j) E(x_k) - \frac{2}{N} N E(x_i) x_T + x_T^2$

$= \frac{E[x_i^2]}{N} + \frac{(N-1)}{N} x_T^2 - 2x_T^2 + x_T^2$

$= \frac{E(x_i)^2}{N} - \frac{x_T^2}{N} \longrightarrow$ (A)

$s^2 = E[(x_i - \bar{x})^2]$

$= E[x_i^2 - 2(x_i)(\bar{x}) + \bar{x}^2]$

$= E\left[x_i^2 - 2(x_i)\sum_j \frac{x_j}{N} + \left(\sum_i \frac{x_i}{N}\right)^2\right]$

$= E[x_i^2] - \frac{2}{N} E\left[(x_i)^2 + \sum_{j \neq i} x_j x_i\right] + \frac{1}{N^2} E\left(\sum_i x_i^2 + \sum_{i \neq j} x_j x_k\right)$

$= E[x_i^2] - \frac{2}{N} E[x_i^2] - \frac{2}{N} E\left[\sum_{j \neq i} x_j x_k\right] + \frac{1}{N^2} E\left[\sum_i x_i^2 + \sum_{i \neq j} x_j x_k\right]$

$= E[x_i^2] - \frac{2}{N} E[x_i^2] - \frac{2}{N}(N-1)x_T^2 + \frac{1}{N^2} N E[x_i] + \frac{1}{N^2} N(N-1) x_T^2$

$= \frac{N-1}{N} E(x_i^2) - \frac{(N-1)x_T^2}{N}$

$= (N-1)\left(\frac{E(x_i^2)}{N} - \frac{x_T^2}{N}\right) \longrightarrow$ (B)

Comparing (A) and (B)

$$s^2 = (N-1)\sigma^2$$

$$s = \sqrt{N-1}\ \sigma \quad Q.E.D$$

Q4 $X \sim N(\mu, \sigma^2)$

$$Y = e^x$$

$$P(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right]$$

$$P(Y \leq y) = P(e^x \leq y) = P(x \leq \ln y)$$

$$P(x \leq \ln y) = \int_0^{\ln y} \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] dx$$

For $\mu = 0$, $\sigma = 1$, $y = 1$

$$P(x \leq \ln y) = \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) dx = \frac{1}{2}.$$

Q5 $P(n;\bar{n}) = \frac{1}{n!} e^{-\bar{n}} \bar{n}^n$

(1) $= \sum_{n=0}^{\infty} \frac{1}{n!} e^{-\bar{n}} \bar{n}^n$

$$= e^{-\bar{n}} \underbrace{\sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!}}_{\text{taylor series expansion}} \Rightarrow e^{-\bar{n}} e^{\bar{n}} = 1. \quad Q.E.D$$

② mean, $\bar{x} = \sum_{n=0}^{\infty} n P(n;\bar{n})$

$= \sum_{n=0}^{\infty} n \frac{1}{n!} e^{-\bar{n}} \bar{n}^{n}$

$= e^{-\bar{n}} \sum_{n=0}^{\infty} \frac{n}{n!} \bar{n}^{n}$

$= e^{-\bar{n}} \sum_{n=0}^{\infty} \frac{\bar{n}^{n}}{(n-1)!}$

$= e^{-\bar{n}} \bar{n} \sum_{n=1}^{\infty} \frac{\bar{n}^{(n-1)}}{(n-1)}$, let $k = n-1$.

$= e^{-\bar{n}} \bar{n} \sum_{k=0}^{\infty} \frac{\bar{n}^{k}}{k!}$

$= e^{-\bar{n}} \bar{n} e^{\bar{n}} \Rightarrow \bar{n}$ Q.E.D

③ <u>Variance</u>,

$= \sum_{n=0}^{\infty} (n-\bar{n})^{2} P(n;\bar{n})$

$= \sum_{n=0}^{\infty} (n^{2} - 2n\bar{n} + \bar{n}^{2}) P(n;\bar{n})$

$= \sum_{n=0}^{\infty} n^{2} P(n;\bar{n}) - 2\bar{n} \sum_{n=0}^{\infty} n P(n;\bar{n}) + \bar{n}^{2} \sum_{n=0}^{\infty} P(n;\bar{n})$

$= \sum_{n=0}^{\infty} n^{2} P(n;\bar{n}) - 2\bar{n}\bar{n} + \bar{n}^{2}$

$= e^{-\bar{n}} . \bar{n} . \sum_{n=1}^{\infty} \frac{n . \bar{n}^{(n-1)}}{(n-1)!} - \bar{n}^{2}$, let $k = n-1$

$= e^{-\bar{n}} \bar{n} . \sum_{k=0}^{\infty} \frac{(k+1) \bar{n}^{k}}{k!} - \bar{n}^{2}$

$= e^{-\bar{n}} \bar{n} \left(\sum_{k=0}^{\infty} \frac{k \bar{n}^{k}}{k!} + \sum_{k=0}^{\infty} \frac{\bar{n}^{k}}{k!} \right) - \bar{n}^{2}$

$= \bar{n}(\bar{n}+1) - \bar{n}^2$

$= \bar{n}$. Q.E.D.

(4). $P(n;\bar{n}) = \frac{1}{n!} e^{-\bar{n}} \bar{n}^{n}$

∴ stirling's formula: $n! = \sqrt{2\pi n}\, e^{-n} n^{n}$

$$\ln P(n;\bar{n}) = \ln\left(\frac{\bar{n}^{n} e^{-\bar{n}}}{n^{n} e^{-n} \sqrt{2\pi n}}\right)$$

$$= n \ln \bar{n} - \bar{n} - n \ln n + n - \ln\sqrt{2\pi n}$$

$$= n \ln\left(\frac{\bar{n}}{n}\right) + (n - \bar{n}) - \ln\sqrt{2\pi n}$$

$\therefore k = n - \bar{n}$

$$= (k+\bar{n}) \ln\left(\frac{\bar{n}}{(k+\bar{n})}\right) + k - \ln\sqrt{2\pi(k+\bar{n})}$$

$$\cancel{= (k+\bar{n}) \ln (1 + k/\bar{n})}$$

$$= (k+\bar{n}) \ln\left(\frac{\bar{n}}{k+\bar{n}}\right) + k - \ln\sqrt{2\pi(k+\bar{n})}$$

$$= (k+\bar{n}) \ln (1 + k/n)^{-1} + k - \ln\sqrt{2\pi(k+\bar{n})}$$

$$= -(k+\bar{n}) \ln (1 + k/n) + k - \ln\sqrt{2\pi(k+\bar{n})}$$

Simplifying first term using wolfram,

$$= -\cancel{k} - \frac{k^2}{2\bar{n}} + \frac{k^3}{6\bar{n}^2} + \cancel{k} - \ln\sqrt{2\pi(k+\bar{n})}$$

$$\ln(P) = -\frac{k^2}{2\bar{n}} + \frac{k^3}{6\bar{n}^2} - \ln\sqrt{2\pi(k+\bar{n})}$$

for large values of $\bar{n}$, we can ignore the second term.

And $k+\bar{n} \approx \bar{n}$

$$P(n) = \frac{\exp\left(\frac{-k^2}{2\bar{n}}\right)}{\sqrt{2\pi n}}$$

$$\therefore k = n - \bar{n}$$

$$P(n) = \frac{\exp\left(-\frac{(n-\bar{n})^2}{2\bar{n}}\right)}{\sqrt{2\pi n}}.$$

For large values of $\bar{n}$, the Poisson distribution approaches a Gaussian distribution, as shown.

In[383]:=

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(*QUESTION 6*)

(*Mean*)
f = (1/2) Sin[x];
xbar = Integrate[f * x, {x, 0, Pi}] // N  (* Mean *)

(* Variance *)
var = Integrate[f * (x - xbar) ^2, {x, 0, Pi}] // N

(*Skewness*)
s = FullSimplify[Integrate[f * (x - xbar) ^3, {x, 0, Pi}]]  // N

(*Kurtosis*)
k = FullSimplify[Integrate[f * (x - xbar) ^4, {x, 0, Pi}]]  // N

sigma = sqrt[var] // N;

(*for same mean and standard dev, skewness and kurtosis for the Normal Distribution*)

(*Skewness for Normal Distribution*)
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S = Integrate[(x - xbar) ^ (3) $\frac{1}{\text{sigma} * \sqrt{2\,\text{Pi}}}$ Exp[$\frac{-(x - \text{xbar})\,^\wedge 2}{2 * \text{sigma}\,^\wedge 2}$], {x, 0, Pi}] // N

(*Kurtosis for Normal Distribution*)

K = Integrate[(x - xbar) ^ (4) $\frac{1}{\text{sigma} * \sqrt{2\,\text{Pi}}}$ Exp[$\frac{-(x - \text{xbar})\,^\wedge 2}{2 * \text{sigma}\,^\wedge 2}$], {x, 0, Pi}] // N

Out[384]= 1.5708

Out[385]= 0.467401

Out[386]= 0.

Out[387]= 0.479255

Out[389]= 0.

Out[390]= $3.\ \text{Erf}\left[\frac{1.11072}{\text{sqrt}[0.467401]}\right] \text{sqrt}[0.467401]^4 + 2.71828^{-\frac{1.2337}{\text{sqrt}[0.467401]^2}} \left(-3.09243\ \text{sqrt}[0.467401] - 3.75994\ \text{sqrt}[0.467401]^3\right)$

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(*While the skewness for both the curves is the same, zero,
the kurtosis for the Normal distribution is greater than that for the given function.*)
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(*QUESTION 7*)

(*For N = 2 *)
list1 = {}; sum = 0;
For[i = 1, i < 500, i++,
 For[j = 1, j < 2, j++,
   sum = sum + RandomVariate[UniformDistribution[{0, 1}]]
  ]
  AppendTo[list1, sum]; sum = 0;
]
Histogram[list1, 20]

(*For N = 8 *)

list2 = {}; sum = 0;
For[i = 1, i < 500, i++,
 For[j = 1, j < 8, j++,
   sum = sum + RandomVariate[UniformDistribution[{0, 1}]]
  ]
  AppendTo[list2, sum]; sum = 0;
]
Histogram[list2, 20]

(*For N = 20 *)

list3 = {}; sum = 0;
For[i = 1, i < 500, i++,
 For[j = 1, j < 20, j++,
   sum = sum + RandomVariate[UniformDistribution[{0, 1}]]
  ]
  AppendTo[list3, sum]; sum = 0;
]
Histogram[list3, 20]

(*For N = 50 *)

list4 = {}; sum = 0;
For[i = 1, i < 500, i++,
 For[j = 1, j < 50, j++,
   sum = sum + RandomVariate[UniformDistribution[{0, 1}]]
  ]
  AppendTo[list4, sum]; sum = 0;
]
Histogram[list4, 20]
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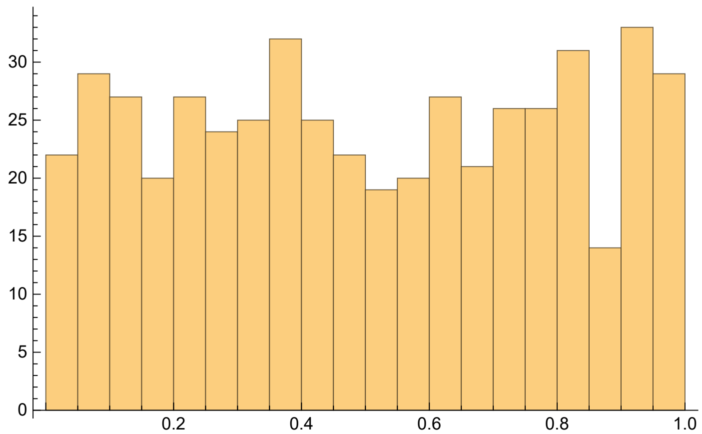
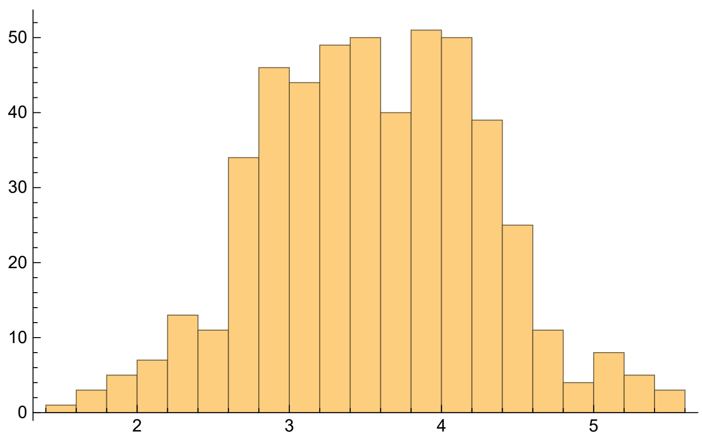
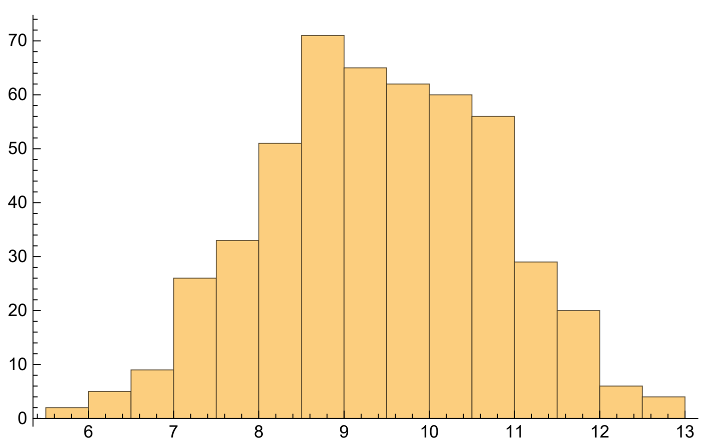
Out[365]=



Out[368]=



Out[371]=



Out[374]=