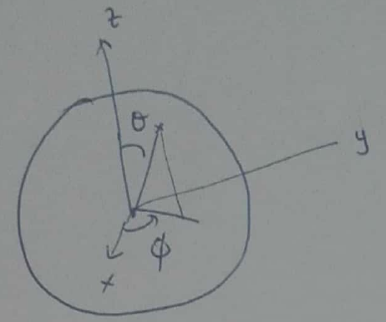
Quiz 1, QM 1

Fall 2019

Q 1.(a)



One needs only two real parameters θ and ϕ to completely characterize the quantum state. The state can be expressed in terms of these parameters:

$$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|S\rangle + \sin\left(\frac{\theta}{2}\right) e^{i\phi}|B\rangle . \quad (1)$$

(b)

$$|\psi\rangle = \left(\frac{1+i}{2}\right)\left(|S\rangle + \left(\frac{-1+i}{1+i}\right)|B\rangle\right)$$

$$= \left(\frac{1+i}{2}\right)\left(|S\rangle + (-1)\left(\frac{1-i}{1+i}\right)|B\rangle\right)$$

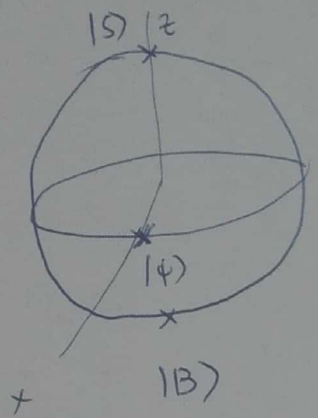
$$= \left(\frac{1+i}{2}\right)\left(|S\rangle - \frac{e^{-i\pi/2}}{e^{i\pi/2}}|B\rangle\right)$$

$$= \left(\frac{1+i}{2}\right)\left(|S\rangle - e^{-i\pi}|B\rangle\right)$$

$$= \left(\frac{1+i}{\sqrt{2}}\right)\left(\frac{|S\rangle + |B\rangle}{\sqrt{2}}\right)$$

Comparing with (1), this yields

$$\theta = \pi/2, \quad \phi = 0.$$



(c) $$|\phi\rangle = \left(\frac{-1+i}{2}\right)\left(|S\rangle + \left(\frac{-1-i}{-1+i}\right)|B\rangle\right)$$

$$= \left(\frac{-1+i}{2}\right)\left(|S\rangle + \left(\frac{1+i}{1-i}\right)|B\rangle\right)$$

$$= \left(\frac{-1+i}{2}\right)\left(|S\rangle + \frac{e^{i\pi}}{e^{-i\pi}}|B\rangle\right)$$

$$= \frac{(-1+i)}{\sqrt{2}}\left(\frac{|S\rangle + |B\rangle}{\sqrt{2}}\right).$$

This is the same as $|\psi\rangle$ upto a global phase factor. Note that the relative phase between $|S\rangle$ and $|B\rangle$ is identical. So $|\psi\rangle$ and $|\phi\rangle$ are physically the same states.

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(d) Let's first find

$$\langle \chi | \psi \rangle = \frac{1}{\sqrt{2}}\left(\langle S| + \langle B|\right)\left(\frac{1+i}{\sqrt{2}}\right)\frac{1}{\sqrt{2}}\left(|S\rangle + |B\rangle\right)$$

$$= \frac{1}{\cancel{4}}\left(\frac{1+i}{\sqrt{2}}\right)\left(\cancel{2}\right) = \frac{1+i}{\sqrt{2}}.$$

$$|\langle \chi|\psi\rangle|^2 = \frac{1}{2}(1-i)(1+i) = \frac{1}{2}\left(1+\cancel{i}-\cancel{i}+1\right) = 1.$$

The desired probability is in fact 1, showing that $|\psi\rangle$ and $|\chi\rangle$ are physically identical.

Q2.

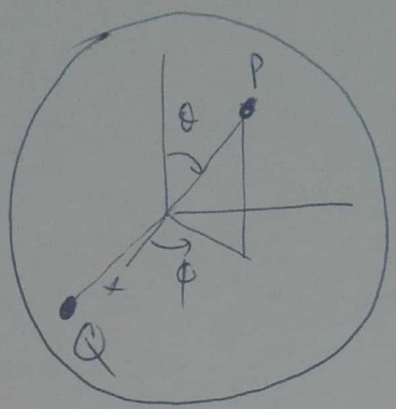
(a) For normalization $|\langle \psi|\psi\rangle|^2 = 1.$

$$\langle \psi|\psi\rangle = \left(\cos\frac{\theta}{2}\langle \alpha| + e^{-i\phi}\sin\frac{\theta}{2}\langle \beta|\right)\left(\cos\frac{\theta}{2}|\alpha\rangle + e^{i\phi}\sin\frac{\theta}{2}|\beta\rangle\right)$$

$$= \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} = 1.$$

Yes, it's normalized.

(b)



$|\psi\rangle$ lives at P on the Bloch sphere. P has polar coordinates (θ, ϕ).

The orthogonal state $|\psi^\perp\rangle$ will be on the diametrically opposite side, i.e. at Q given by $(\pi-\theta, \phi+\pi)$. So the orthogonal state is (~~ϕ~~ replacing θ by $\pi-\theta$ and ϕ by $\phi+\pi$) in $|\psi\rangle$ yields

$$|\psi^\perp\rangle = \cos\left(\frac{\pi-\theta}{2}\right)|\alpha\rangle + e^{i(\phi+\pi)}\sin\left(\frac{\pi-\theta}{2}\right)|\beta\rangle$$

since $\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right) = \sin\frac{\theta}{2}$,

$$\sin\left(\frac{\pi}{2}-\frac{\theta}{2}\right) = \cos\frac{\theta}{2}$$

$$\begin{aligned} e^{i(\phi+\pi)} &= \cos(\phi+\pi)+i\sin(\phi+\pi) \\ &= -\cos\phi - i\sin\phi \\ &= -(\cos\phi+i\sin\phi) \\ &= -e^{i\phi} \end{aligned}$$

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$\therefore$

$$\boxed{|\psi^{\perp}\rangle = \sin\frac{\theta}{2}|\alpha\rangle - e^{i\phi}\cos\frac{\theta}{2}|\beta\rangle.}$$

Check

$$\langle\psi^{\perp}|\psi\rangle = \left(\sin\frac{\theta}{2}\langle\alpha| - e^{-i\phi}\cos\frac{\theta}{2}\langle\beta|\right)\left(\cos\frac{\theta}{2}|\alpha\rangle + e^{i\phi}\sin\frac{\theta}{2}|\beta\rangle\right)$$

$$= \sin\frac{\theta}{2}\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\cos\frac{\theta}{2} = 0 \text{ as desired.}$$

(c)

$$\langle\phi|\psi\rangle = \frac{1}{\sqrt{2}}\left(\langle\alpha| + \langle\beta|\right)\left(\cos\frac{\theta}{2}|\alpha\rangle + e^{i\phi}\sin\frac{\theta}{2}|\beta\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\cos\frac{\theta}{2} + e^{i\phi}\sin\frac{\theta}{2}\right)$$

The probability is

$$|\langle\phi|\psi\rangle|^2 = \frac{1}{2}\left(\cos\frac{\theta}{2} + e^{-i\phi}\sin\frac{\theta}{2}\right)\left(\cos\frac{\theta}{2} + e^{i\phi}\sin\frac{\theta}{2}\right)$$

$$= \frac{1}{2}\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} + e^{i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2} + e^{-i\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{\cancel{2}} \sin\theta \cdot \cancel{2}\cos\phi\right)$$

$$= \frac{1}{2}\left(1 + \sin\theta\cos\phi\right).$$

Q3.

$$p_{+x} = \left|\langle x|\psi\rangle\right|^2 = \frac{3}{4}$$

This means

$$\frac{3}{4} = \left|\left(\frac{1}{\sqrt{2}}\langle\alpha| + \frac{1}{\sqrt{2}}\langle\beta|\right)\left(\frac{1}{\sqrt{2}}|\alpha\rangle + e^{i\phi}\frac{1}{\sqrt{2}}|\beta\rangle\right)\right|^2$$

$$= \left|\frac{1}{2} + \frac{1}{2}e^{i\phi}\right|^2$$

$$= \frac{1}{4}\left(1+e^{-i\phi}\right)\left(1+e^{i\phi}\right) = \frac{1}{4}\left(2 + 2\cos\phi\right)$$

$$= \frac{1}{2}\left(1+\cos\phi\right) = \cos^2\frac{\phi}{2}.$$

Similarly

$$p_{-x} = \frac{1}{4} = \left|\langle -x|\psi\rangle\right|^2$$

$$= \left|\left(\frac{1}{\sqrt{2}}\langle\alpha| - \frac{1}{\sqrt{2}}\langle\beta|\right)\left(\frac{1}{\sqrt{2}}|\alpha\rangle + e^{i\phi}\frac{1}{\sqrt{2}}|\beta\rangle\right)\right|^2$$

$$= \left|\frac{1}{2} - \frac{1}{2}e^{i\phi}\right|^2$$

$$= \frac{1}{4}\left(1 - e^{-i\phi}\right)\left(1 - e^{i\phi}\right)$$

$$= \frac{1}{4}\left(1 + 1 - e^{i\phi} - e^{-i\phi}\right)$$

$$= \frac{1}{4}\left(2 - 2\cos\phi\right) = \frac{1}{2}\left(1 - \cos\phi\right) = \sin^2\frac{\phi}{2}.$$

$$P_{+y} = \left|\langle y|\psi\rangle\right|^2$$

$$= \left|\left(\frac{1}{\sqrt{2}}\langle\alpha| - \frac{i}{\sqrt{2}}\langle\beta|\right)\left(\frac{1}{\sqrt{2}}|\alpha\rangle + e^{i\phi}\frac{1}{\sqrt{2}}|\beta\rangle\right)\right|^2$$

$$= \left|\left(\frac{1}{2} - \frac{i}{2}e^{i\phi}\right)\right|^2$$

$$= \frac{1}{4}\left(1 + ie^{-i\phi}\right)\left(1 - ie^{i\phi}\right)$$

$$= \frac{1}{4}\left(1 + 1 - ie^{i\phi} + ie^{-i\phi}\right)$$

$$= \frac{1}{4}\left(2 - i\cdot 2i\sin\phi\right)$$

$$= \frac{1}{2}\left(1 + \sin\phi\right)$$

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$$P_{-y} = |\langle -y|\psi\rangle|^2$$

$$= \left|\left(\frac{1}{\sqrt{2}}\langle\alpha| + \frac{i}{\sqrt{2}}\langle\beta|\right)\left(\frac{|\alpha\rangle}{\sqrt{2}} + \frac{e^{i\phi}}{\sqrt{2}}|\beta\rangle\right)\right|^2$$

$$= \left|\frac{1}{2} + \frac{i}{2}e^{i\phi}\right|^2$$

$$= \frac{1}{4}\left(1 + ie^{i\phi}\right)\left(1 - ie^{-i\phi}\right)$$

$$= \frac{1}{4}\left(1 - ie^{-i\phi} + ie^{i\phi} + 1\right)$$

$$= \frac{1}{4}\left(2 + i \cdot 2i\sin\phi\right)$$

$$= \frac{1}{2}\left(1 - \sin\phi\right).$$

Let's summarize:

$$P_{+x} = \frac{3}{4} = \cos^2\phi/2 \quad \text{—} (1)$$

$$P_{-x} = \frac{1}{4} = \sin^2\phi/2 \quad \text{—} (2)$$

(2) and (1) provide the same information; so one needs only one of them.

$$P_{+y} = \frac{1}{2}(1 + \sin\phi) = \frac{1}{15} \quad \text{—} (3)$$

$$P_{-y} = \frac{1}{2}(1 - \sin\phi) = \frac{14}{15} \quad \text{—} (4)$$

Similarly (3) and (4) provide the same information.

From (1):

$$\cos\frac{\phi}{2} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \frac{\phi}{2} = \pm\frac{\pi}{6} \quad (30^\circ)$$

$$\phi = +\frac{\pi}{3} \text{ or } -\frac{\pi}{3}.$$

We need to choose whether $\phi = \frac{\pi}{3}$ or $-\frac{\pi}{3}$. For that let's use (3):

$$1 + \sin\phi = \frac{2}{15}$$

$$\sin\phi = \frac{2}{15} - \frac{15}{15}$$

$$= -\frac{13}{15}$$

$$\phi = \sin^{-1}\left(-13/15\right) \cancel{= -\frac{\pi}{3}}$$

which sets ϕ at $-\frac{\pi}{3}$ (-60°).