Quiz 1
Time: 1 Hour

1. **Sweet Electrons (10 Points)**

   The PhysLab team discovers a hitherto unknown property of electrons and chooses to call it "sweetness." Every electron observed is either sweet or bitter; therefore, the team maintains that sweetness is a binary quantum property, one which could be studied like spin. The state of a sweet electron is represented by $|S\rangle$ whereas that of a bitter electron is represented by $|B\rangle$.

   (a) **(2 Points)** Argue that one needs only two real parameters to completely specify the sweetness-state of an electron.
   Hint: Think of the Bloch sphere.
   (b) **(4 Points)** An electron has been prepared in the following state:

   $$|\psi\rangle = \left(\frac{1+i}{2}\right)|S\rangle + \left(\frac{-1+i}{2}\right)|B\rangle.$$

   Represent this state on the Bloch sphere.
   (c) **(2 Points)** Is $|\psi\rangle$ different from

   $$|\phi\rangle = \left(\frac{-1+i}{2}\right)|S\rangle + \left(\frac{-1-i}{2}\right)|B\rangle?$$

   Give reason(s) for your answer.
   (d) **(2 Points)** What is the probability that an electron prepared in state $|\psi\rangle$ is measured in the state

   $$|\chi\rangle = \frac{1}{\sqrt{2}}|S\rangle + \frac{1}{\sqrt{2}}|B\rangle?$$

2. **Not so Sweet Spins (10 Points)**

   The spin-state of an electron could also be mapped to the Bloch sphere. An electron is prepared in spin-state

   $$|\psi\rangle = \cos\left(\frac{\theta}{2}\right)|\alpha\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|\beta\rangle,$$

   where $|\alpha\rangle$ and $|\beta\rangle$ denote, respectively, the spin-up and spin-down states with respect to the applied magnetic field.

(a) **(2 Points)** Check if this state is normalized.

(b) **(4 Points)** Find a state that is orthogonal to $|\psi\rangle$. Make sure that the state you write is also normalized.

Hint: Represent both $|\psi\rangle$ and the state orthogonal to it on the Bloch sphere, and see how the parameters specifying the latter relate to those specifying the former.

(c) **(4 Points)** What is the probability that the electron prepared in state $|\psi\rangle$ would be measured in the state

$$|\phi\rangle = \frac{1}{\sqrt{2}}|\alpha\rangle + \frac{1}{\sqrt{2}}|\beta\rangle?$$

3. **The Mystery State (5 Points)**

During our experiments on sweetness and spin, we ended up preparing some electrons in an unknown (but interesting) state. We know that the state has the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}|\alpha\rangle + e^{i\phi}\frac{1}{\sqrt{2}}|\beta\rangle,$$

but we do not know the value of ϕ as yet. Being busy working on other things in PhysLab, we delegate the (mundane) task of determining ϕ to you. Measuring the spin components of the electrons in state $|\psi\rangle$, we obtained the following probabilities:

$$p_{+x} = \frac{3}{4} \qquad p_{+y} = \frac{1}{15}$$
$$p_{-x} = \frac{1}{4} \qquad p_{-y} = \frac{14}{15}$$

Use them to calculate ϕ. Note that

$$|+x\rangle = \frac{1}{\sqrt{2}}|\alpha\rangle + \frac{1}{\sqrt{2}}|\beta\rangle,$$

$$|-x\rangle = \frac{1}{\sqrt{2}}|\alpha\rangle - \frac{1}{\sqrt{2}}|\beta\rangle,$$

$$|+y\rangle = \frac{1}{\sqrt{2}}|\alpha\rangle + i\frac{1}{\sqrt{2}}|\beta\rangle,$$

and

$$|-y\rangle = \frac{1}{\sqrt{2}}|\alpha\rangle - i\frac{1}{\sqrt{2}}|\beta\rangle.$$