Quantum Mechanics

Assignment 1

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1. The polarization of a photon is a two level quantum system. The plane of polarization is specified as a quantum state

$$|\psi\rangle = C_H|H\rangle + C_V|V\rangle$$

where $|H\rangle$ and $|V\rangle$ correspond to one possible basis and physically represent horizontal and vertical polarization

i.e $|H\rangle = |\theta = 0\rangle$ and $|V\rangle = |\theta = \pi/2\rangle$.

Suppose $|\theta\rangle$ represents the state of a beam of photons linearly polarized at an angle of θ from the horizontal

a) Write $|\theta\rangle$ as a linear combination of $|H\rangle$ and $|V\rangle$

Ans $$|\theta\rangle = \cos\theta|H\rangle + \sin\theta|V\rangle$$

b) What is the probability that a photon in the state $|\theta\rangle$ will

be measured to have vertical polarization?

Ans $\langle V|\theta\rangle = \langle V|(\cos\theta|H\rangle + \sin\theta|V\rangle)$

$= \cos\theta\langle V|H\rangle + \sin\theta\langle V|V\rangle$

$= \sin\theta$

$P(V||\theta\rangle) = |\langle V|\theta\rangle|^2$

$= \sin^2\theta$

c. What is the probability that a photon in the state $|\theta\rangle$, will be measured to have linear polarization along $+45°$?

$|45°\rangle = \frac{1}{\sqrt{2}}(|H\rangle + |V\rangle)$

$\langle 45°| = \frac{1}{\sqrt{2}}(\langle H| + \langle V|)$

$= \frac{1}{\sqrt{2}}\langle H| + \frac{1}{\sqrt{2}}\langle V|$

$|\theta\rangle = \cos\theta|H\rangle + \sin\theta|V\rangle$

$\langle 45°|\theta\rangle = \left(\frac{1}{\sqrt{2}}\langle H| + \frac{1}{\sqrt{2}}\langle V|\right)\left(\cos\theta|H\rangle + \sin\theta|V\rangle\right)$

$= \frac{1}{\sqrt{2}}\cos\theta\langle H|H\rangle + \frac{1}{\sqrt{2}}\sin\theta\langle H|V\rangle + \frac{1}{\sqrt{2}}\cos\theta\langle V|H\rangle + \frac{1}{\sqrt{2}}\sin\theta\langle V|V\rangle$

$\langle H|H\rangle = 1$ $\langle V|V\rangle = 1$ $\langle V|H\rangle = 0$ $\langle H|V\rangle = 0$

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$$= \frac{1}{\sqrt{2}} \cos\theta + \frac{1}{\sqrt{2}} \sin\theta$$

$$P(+45 | |\theta\rangle) = |\langle +45 | \theta\rangle|^2$$

$$= \left| \left(\frac{1}{\sqrt{2}} \cos\theta + \frac{1}{\sqrt{2}} \sin\theta \right) \right|^2$$

$$= \frac{1}{2} (\cos\theta + \sin\theta)^2$$

$$= \frac{1}{2} (\cos^2\theta + \sin^2\theta + 2\cos\theta \sin\theta)$$

$$= \frac{1}{2} (1 + \sin 2\theta)$$

d) What is the probability that a photon in the state $|\theta\rangle$, will be measured to have right circular polarization, which is given by $\frac{1}{\sqrt{2}} (|H\rangle - i|V\rangle)$

Ans $$|R\rangle = \frac{1}{\sqrt{2}} (|H\rangle - i|V\rangle)$$

$$\langle R| = \frac{1}{\sqrt{2}} (\langle H| + i\langle V|)$$

$$= \frac{1}{\sqrt{2}} \langle H| + \frac{i}{\sqrt{2}} \langle V|$$

$$|\theta\rangle = \cos\theta |H\rangle + \sin\theta |V\rangle$$

$$\langle R|\theta\rangle = \left(\frac{1}{\sqrt{2}} \langle H| + \frac{i}{\sqrt{2}} \langle V| \right) (\cos\theta |H\rangle + \sin\theta |V\rangle)$$

$$= \frac{1}{\sqrt{2}} \cos\theta \langle H|H\rangle + \frac{1}{\sqrt{2}} \sin\theta \langle H|V\rangle + \frac{1}{\sqrt{2}} i \cos\theta \langle V|H\rangle + \frac{i}{\sqrt{2}} \sin\theta \langle V|V\rangle$$

$\langle H|H\rangle = 1$ $\quad \langle V|V\rangle = 1$ $\quad \langle H|V\rangle = 0$ $\quad \langle V|H\rangle = 0$

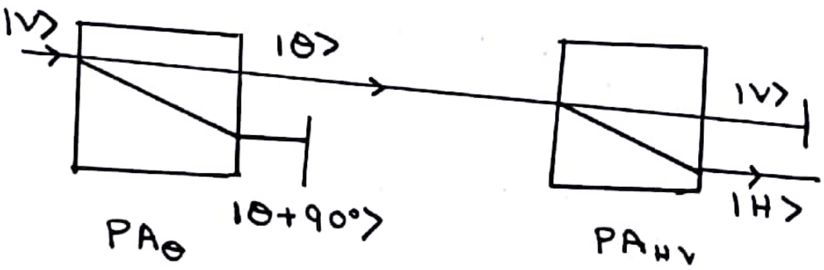
$$= \frac{1}{\sqrt{2}} \cos\theta + \frac{i}{\sqrt{2}} \sin\theta = \frac{1}{\sqrt{2}} e^{i\theta}$$

$$P(R||\theta\rangle) = |\langle R|\theta\rangle|^2$$

$$= \left(\frac{1}{\sqrt{2}} e^{i\theta}\right)\left(\frac{1}{\sqrt{2}} e^{-i\theta}\right)$$

$$= \frac{1}{2}$$

e) If a beam of photon in state $|V\rangle$ is sent through a series of two polarization beam splitter (polarization analyzer), as illustrated in Fig (1) then



(i) What fraction of input photon will survive to the final output?

Ans $|\theta\rangle = \cos\theta |H\rangle + \sin\theta |V\rangle$

$$|\theta + 90^\circ\rangle = \cos(\theta + 90^\circ)|H\rangle + \sin(\theta + 90^\circ)|V\rangle$$

$$|\theta+90^\circ\rangle = -\sin\theta|H\rangle + \cos\theta|V\rangle$$

$$\langle\theta|V\rangle = (\cos\theta\langle H| + \sin\theta\langle V|)|V\rangle$$

$$= \cos\theta\langle H|V\rangle + \sin\theta\langle V|V\rangle$$

$$= \sin\theta$$

$$\langle H|\theta\rangle = \langle H|(\cos\theta|H\rangle + \sin\theta|V\rangle)$$

$$= \cos\theta\langle H|H\rangle + \sin\theta\langle H|V\rangle$$

$$= \cos\theta$$

$$P(\theta||V\rangle) = |\langle\theta|V\rangle|^2$$

$$= \sin^2\theta$$

$$P(H||\theta\rangle) = |\langle H|\theta\rangle|^2$$

$$= \cos^2\theta$$

Fraction of photon after $PA_\theta = \sin^2\theta$

Fraction of photon after $PA_{HV} = \cos^2\theta$

Fraction being transmitted $= \sin^2\theta\cos^2\theta$

ii At what angle θ must the PA_θ be oriented so as to maximize the number of photons that are transmitted

by the PA_{HV}? What fraction of the photons are transmitted for this particular value of θ?

$\sin^2\theta \cos^2\theta \Rightarrow$ maximum

$$\frac{4\sin^2\theta\cos^2\theta}{4} = \frac{(\sin 2\theta)^2}{4}$$

$$\sin 2\theta = 1$$

$$2\theta = 90^\circ$$

$$\theta = 45^\circ$$

$$\theta = \frac{\pi}{4}$$

The fraction transmitted $= \frac{1}{4}$

iii) What fraction of photons are transmitted if the PA_θ is simply removed from the experiment

Ans All the photons will be vertically polarized and will be blocked

$$\langle H|V\rangle = 0$$

$$P(H|\,|V\rangle) = |\langle H|V\rangle|^2$$

$$= 0$$

2. Consider the following unnormalized state vector of a 2 dimensional quantum system

$$|\psi_1\rangle = 3|+\rangle + 4|-\rangle$$

$$|\psi_2\rangle = |+\rangle + 2i|-\rangle$$

$$|\psi_3\rangle = 3|+\rangle - e^{i\pi/3}|-\rangle$$

where $|+\rangle$ and $|-\rangle$ represent spins pointing along (anti along) the x direction

a) Normalize each state vector.

Ans $$|\psi_1\rangle = 3|+\rangle + 4|-\rangle$$

$$|\psi_1\rangle = c[3|+\rangle + 4|-\rangle]$$

c = complex multiplicative factor

$$\langle\psi_1| = c^*[3\langle+| + 4\langle-|]$$

$$\langle\psi_1|\psi_1\rangle = \left(c^*[3\langle+| + 4\langle-|]\right)\left(c[3|+\rangle + 4|-\rangle]\right)$$

$$1 = c^2\left(3\times3\langle+|+\rangle + 12\langle+|-\rangle + 12\langle-|+\rangle + 16\langle-|-\rangle\right)$$

$$1 = c^2(9+16)$$

$$c^2 = \frac{1}{25}$$

$$c = \frac{1}{5}$$ (we choose c to be real) and positive

$$|\psi_1\rangle = \frac{3}{5}|+\rangle + \frac{4}{5}|-\rangle$$

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$$|\psi_2\rangle = |+\rangle + 2i\,|-\rangle$$

$$|\psi_2\rangle = C_2\,[|+\rangle + 2i|-\rangle]$$

$$\langle\psi_2| = C_2^*\,[\langle+| - 2i\,\langle-|]$$

$$\langle\psi_2|\psi_2\rangle = \left(C_2^*[\langle+| - 2i\langle-|]\right)\left(C_2[|+\rangle + 2i\,|-\rangle]\right)$$

$$1 = C_2^2[\,\langle+|+\rangle + 2i\,\langle+|-\rangle - 2i\,\langle-|+\rangle + 4\,\langle-|-\rangle]$$

$$1 = C_2^2\,[1+4]$$

$$C_2^2 = \frac{1}{5}$$

$$C_2 = \frac{1}{\sqrt{5}}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{5}}\,[|+\rangle + 2i\,|-\rangle]$$

$$= \frac{1}{\sqrt{5}}|+\rangle + \frac{2i}{\sqrt{5}}|-\rangle \quad \checkmark$$

$$|\psi_3\rangle = 3|+\rangle - e^{i\pi/3}\,|-\rangle$$

$$|\psi_3\rangle = C\,[3|+\rangle - e^{i\pi/3}\,|-\rangle]$$

$$\langle\psi_3| = C^*\,[3\langle+| - e^{-i\pi/3}\,\langle-|]$$

$$\langle\psi_3|\psi_3\rangle = C^*[3\langle+| - e^{-i\pi/3}\langle-|]\,C\,[3|+\rangle - e^{i\pi/3}\,|-\rangle]$$

$$1 = C^2[\,9\,\langle+|+\rangle - 3e^{i\pi/3}\,\langle+|-\rangle - 3e^{-i\pi/3}\,\langle-|+\rangle + \langle-|-\rangle$$

$$1 = C^2\,(10)$$

$$C = \frac{1}{\sqrt{10}} \quad \checkmark$$

$$|\psi_3\rangle = \frac{3}{\sqrt{10}}\,|+\rangle - \frac{e^{i\pi/3}}{\sqrt{10}}\,|-\rangle$$

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b) For each state, calculate the probability that the spin component is up or down along each of the three Cartesian axis. Use bra-ket notation for the entire calculation

Ans

$$P_{1,+} = |\langle + | \psi_1 \rangle|^2$$

$$= \left| \langle + | \left(\frac{3}{5} |+\rangle + \frac{4}{5} |-\rangle \right) \right|^2$$

$$= \left| \frac{3}{5} \langle + | + \rangle + \frac{4}{5} \langle + | - \rangle \right|^2$$

$$= \left| \frac{3}{5} \right|^2 = \frac{9}{25}$$

$$P_{1,-} = |\langle - | \psi_1 \rangle|^2$$

$$= \left| \langle - | \left(\frac{3}{5} |+\rangle + \frac{4}{5} |-\rangle \right) \right|^2$$

$$= \left| \frac{3}{5} \langle - | + \rangle + \frac{4}{5} \langle - | - \rangle \right|^2$$

$$= \left| \frac{4}{5} \right|^2 = \frac{16}{25}$$

$$|+\rangle_x = \frac{1}{\sqrt{2}} \left[|+\rangle + |-\rangle \right]$$

$$_x\langle + | = \frac{1}{\sqrt{2}} \left[\langle + | + \langle - | \right]$$

$$P_{1,+x} = |{}_x\langle + | \psi_1 \rangle|^2$$

$$= \left| \left(\frac{1}{\sqrt{2}} \langle + | + \frac{1}{\sqrt{2}} \langle - | \right) \left(\frac{3}{5} |+\rangle + \frac{4}{5} |-\rangle \right) \right.$$

$$= \left| \frac{3}{5\sqrt{2}} \langle +|+\rangle + \frac{4}{5\sqrt{2}} \langle +|-\rangle + \frac{3}{5\sqrt{2}} \langle -|+\rangle + \frac{4}{5\sqrt{2}} \langle -|-\rangle \right|^2$$

$$= \left(\frac{3}{5\sqrt{2}} + \frac{4}{5\sqrt{2}} \right)^2$$

$$= \left(\frac{7}{5\sqrt{2}} \right)^2 = \frac{49}{50}$$

$$P_{1,-x} = |{}_x\langle - | \psi_1 \rangle|^2$$

$${}_x\langle -| = \frac{1}{\sqrt{2}} [\langle +| - \langle -|]$$

$$= \frac{1}{\sqrt{2}} \langle +| - \frac{1}{\sqrt{2}} \langle -|$$

$$P_{1,-x} = \left| \left(\frac{1}{\sqrt{2}} \langle +| - \frac{1}{\sqrt{2}} \langle -| \right) \left(\frac{3}{5} |+\rangle + \frac{4}{5} |-\rangle \right) \right|^2$$

$$= \left| \frac{3}{5\sqrt{2}} \langle +|+\rangle + \frac{4}{5\sqrt{2}} \langle +|-\rangle - \frac{3}{5\sqrt{2}} \langle -|+\rangle - \frac{4}{5\sqrt{2}} \langle -|-\rangle \right|^2$$

$$= \left| \frac{3}{5\sqrt{2}} - \frac{4}{5\sqrt{2}} \right|^2 = \left| -\frac{1}{5\sqrt{2}} \right|^2 = \frac{1}{50}$$

$$|+\rangle_y = \frac{1}{\sqrt{2}} [|+\rangle + i|-\rangle]$$

$${}_y\langle +| = \frac{1}{\sqrt{2}} \langle +| - \frac{i}{\sqrt{2}} \langle -|$$

$$|\psi_1\rangle = \frac{3}{5} |+\rangle + \frac{4}{5} |-\rangle$$

$$P_{1,+y} = |{}_y\langle + | \psi_1 \rangle|^2$$

$$= \left| \left(\frac{1}{\sqrt{2}} \langle +| - \frac{i}{\sqrt{2}} \langle -| \right) \left(\frac{3}{5} |+\rangle + \frac{4}{5} |-\rangle \right) \right|^2$$

$$= \left| \frac{3}{5\sqrt{2}} \langle +|+\rangle + \frac{4}{5\sqrt{2}} \langle +|-\rangle - \frac{3i}{5\sqrt{2}} \langle -|+\rangle - \frac{4i}{5\sqrt{2}} \langle -|-\rangle \right|^2$$

$$= \left| \frac{3}{5\sqrt{2}} - \frac{4i}{5\sqrt{2}} \right|^2$$

$$= \left(\frac{3}{5\sqrt{2}} - \frac{4i}{5\sqrt{2}} \right) \left(\frac{3}{5\sqrt{2}} + \frac{4i}{5\sqrt{2}} \right)$$

$$= \frac{9}{50} + \frac{12i}{50} - \frac{12i}{50} + \frac{16}{50}$$

$$= \frac{25}{50} = \frac{1}{2}$$

$$|-\rangle_y = \frac{1}{\sqrt{2}} \left[|+\rangle - i|-\rangle \right]$$

$${}_y\langle -| = \frac{1}{\sqrt{2}} \left[\langle +| + i \langle -| \right]$$

$$= \frac{1}{\sqrt{2}} \langle +| + \frac{i}{\sqrt{2}} \langle -|$$

$$|\psi_1\rangle = \frac{3}{5} |+\rangle + \frac{4}{5} |-\rangle$$

$$P_{1,-y} = |{}_y\langle -|\psi_1\rangle|^2$$

$$= \left| \left(\frac{1}{\sqrt{2}} \langle +| + \frac{i}{\sqrt{2}} \langle -| \right) \left(\frac{3}{5} |+\rangle + \frac{4}{5} |-\rangle \right) \right|^2$$

$$= \left| \frac{3}{5\sqrt{2}} \langle +|+\rangle + \frac{4}{5\sqrt{2}} \langle +|-\rangle + \frac{3i}{5\sqrt{2}} \langle -|+\rangle + \frac{4i}{5\sqrt{2}} \langle -|-\rangle \right|^2$$

$$= \left| \frac{3}{5\sqrt{2}} + \frac{4i}{5\sqrt{2}} \right|^2$$

$$= \left(\frac{3+4i}{5\sqrt{2}} \right) \left(\frac{3-4i}{5\sqrt{2}} \right) = \frac{9 - 12i + 12i + 16}{50}$$

$$= \frac{25}{50} = \frac{1}{2}$$

The probabilities for state 2 are

$$|\psi_2\rangle = \frac{1}{\sqrt{5}}|+\rangle + \frac{2i}{\sqrt{5}}|-\rangle$$

$$P_{2,+} = \left|\langle +|\left(\frac{1}{\sqrt{5}}|+\rangle + \frac{2i}{\sqrt{5}}|-\rangle\right)\right|^2$$

$$= \left|\frac{1}{\sqrt{5}}\langle +|+\rangle + \frac{2i}{\sqrt{5}}\langle +|-\rangle\right|^2$$

$$= \frac{1}{5}$$

$$P_{2,-} = \left|\langle -|\left(\frac{1}{\sqrt{5}}|+\rangle + \frac{2i}{\sqrt{5}}|-\rangle\right)\right|^2$$

$$= \left|\frac{1}{\sqrt{5}}\langle -|+\rangle + \frac{2i}{\sqrt{5}}\langle -|-\rangle\right|^2$$

$$= \left|\frac{2i}{\sqrt{5}}\right|^2 = \frac{4}{5}$$

$$P_{2,x} = |{}_x\langle +|\psi_2\rangle|^2$$

$$= \left|\left(\frac{1}{\sqrt{2}}\langle +| + \frac{1}{\sqrt{2}}\langle -|\right)\left(\frac{1}{\sqrt{5}}|+\rangle + \frac{2i}{\sqrt{5}}|-\rangle\right)\right|^2$$

$$= \left|\frac{1}{\sqrt{10}}\langle +|+\rangle + \frac{2i}{\sqrt{10}}\langle +|-\rangle + \frac{1}{\sqrt{10}}\langle -|+\rangle + \frac{2i}{\sqrt{10}}\langle -|-\rangle\right|^2$$

$$= \left(\frac{1+2i}{\sqrt{10}}\right)^2$$

$$= \left(\frac{1+2i}{\sqrt{10}}\right)\left(\frac{1-2i}{\sqrt{10}}\right)$$

$$= \frac{1+4 \ -2i+2i}{\sqrt{10}\sqrt{10}}$$

$$= \frac{5}{10} = \frac{1}{2}$$

$$|-\rangle_x = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle$$

$$_x\langle -| = \frac{1}{\sqrt{2}}\langle +| - \frac{1}{\sqrt{2}}\langle -|$$

$$|\psi_2\rangle = \frac{1}{\sqrt{5}}|+\rangle + \frac{2i}{\sqrt{5}}|-\rangle$$

$$P_{2,-x} = |{}_x\langle -|\psi_2\rangle|^2$$

$$= \left|\left(\frac{1}{\sqrt{2}}\langle +| - \frac{1}{\sqrt{2}}\langle -|\right)\left(\frac{1}{\sqrt{5}}|+\rangle + \frac{2i}{\sqrt{5}}|-\rangle\right)\right|^2$$

$$= \left|\frac{1}{\sqrt{10}}\langle +|+\rangle + \frac{2i}{\sqrt{10}}\langle +|-\rangle - \frac{1}{\sqrt{10}}\langle -|+\rangle - \frac{2i}{\sqrt{10}}\langle -|-\rangle\right|^2$$

$$= \left|\frac{1}{\sqrt{10}} - \frac{2i}{\sqrt{10}}\right|^2$$

$$= \left(\frac{1-2i}{\sqrt{10}}\right)\left(\frac{1+2i}{\sqrt{10}}\right)$$

$$= \frac{1+4 \ +2i-2i}{10} = \frac{5}{10} = \frac{1}{2}$$

$$|+\rangle_y = \frac{1}{\sqrt{2}}\left[|+\rangle + i|-\rangle\right]$$

$$_y\langle +| = \frac{1}{\sqrt{2}}\left[\langle +| - i\langle -|\right]$$

$${}_y\langle +| = \frac{1}{\sqrt{2}}\langle +| - \frac{i}{\sqrt{2}}\langle -|$$

$$|\psi_2\rangle = \frac{1}{\sqrt{5}}|+\rangle + \frac{2i}{\sqrt{5}}|-\rangle$$

$$P_{2,+y} = |{}_y\langle +|\psi_2\rangle|^2$$

$$= \left|\left(\frac{1}{\sqrt{2}}\langle +| - \frac{i}{\sqrt{2}}\langle -|\right)\left(\frac{1}{\sqrt{5}}|+\rangle + \frac{2i}{\sqrt{5}}|-\rangle\right)\right|^2$$

$$= \left|\frac{1}{\sqrt{10}}\langle +|+\rangle + \frac{2i}{\sqrt{10}}\langle +|-\rangle - \frac{i}{\sqrt{10}}\langle -|+\rangle + \frac{2}{\sqrt{10}}\langle -|-\rangle\right|^2$$

$$= \left|\frac{3}{\sqrt{10}}\right|^2 = \frac{9}{10}$$

$$|-\rangle_y = \frac{1}{\sqrt{2}}|+\rangle - \frac{i}{\sqrt{2}}|-\rangle$$

$${}_y\langle -| = \frac{1}{\sqrt{2}}\langle +| + \frac{i}{\sqrt{2}}\langle -|$$

$$P_{2,-y} = |{}_y\langle -|\psi_2\rangle|^2$$

$$= \left|\left(\frac{1}{\sqrt{2}}\langle +| + \frac{i}{\sqrt{2}}\langle -|\right)\left(\frac{1}{\sqrt{5}}|+\rangle + \frac{2i}{\sqrt{5}}|-\rangle\right)\right|^2$$

$$= \left|\left(\frac{1}{\sqrt{10}}\langle +|+\rangle - \frac{2i}{\sqrt{10}}\langle +|-\rangle + \frac{i}{\sqrt{10}}\langle -|+\rangle - \frac{2}{\sqrt{10}}\langle -|-\rangle\right)\right|^2$$

$$= \left|\frac{1}{\sqrt{10}} - \frac{2}{\sqrt{10}}\right|^2 = \left|-\frac{1}{\sqrt{10}}\right|^2 = \frac{1}{10}$$

The probabilities for state 3 are

$$|\psi_3\rangle = \frac{3}{\sqrt{10}}|+\rangle - \frac{e^{i\pi/3}}{\sqrt{10}}|-\rangle$$

$$P_{3,+} = |\langle + | \psi_3 \rangle|^2$$

$$= \left| \langle + | \left(\frac{3}{\sqrt{10}} |+\rangle - \frac{e^{i\pi/3}}{\sqrt{10}} |-\rangle \right) \right|^2$$

$$= \left| \frac{3}{\sqrt{10}} \langle + | + \rangle - \frac{e^{i\pi/3}}{\sqrt{10}} \langle + | - \rangle \right|^2$$

$$= \left| \frac{3}{\sqrt{10}} \right|^2 = \frac{9}{10}$$

$$P_{3,-} = |\langle - | \psi_3 \rangle|^2$$

$$= \left| \langle - | \left(\frac{3}{\sqrt{10}} |+\rangle - \frac{e^{i\pi/3}}{\sqrt{10}} |-\rangle \right) \right|^2$$

$$= \left| \frac{3}{\sqrt{10}} \langle - | + \rangle - \frac{e^{i\pi/3}}{\sqrt{10}} \langle - | - \rangle \right|^2$$

$$= \left| - \frac{e^{i\pi/3}}{\sqrt{10}} \right|^2$$

$$= \left(- \frac{e^{i\pi/3}}{\sqrt{10}} \right) \left(- \frac{e^{-i\pi/3}}{\sqrt{10}} \right)$$

$$= \frac{1}{10}$$

$${}_x\langle + | = \frac{1}{\sqrt{2}} \langle + | + \frac{1}{\sqrt{2}} \langle - |$$

$$P_{3,+x} = |{}_x\langle + | \psi_3 \rangle|^2$$

$$= \left| \left(\frac{1}{\sqrt{2}} \langle + | + \frac{1}{\sqrt{2}} \langle - | \right) \left(\frac{3}{\sqrt{10}} |+\rangle - \frac{e^{i\pi/3}}{\sqrt{10}} |-\rangle \right) \right|^2$$

$$= \left| \frac{3}{\sqrt{20}} \langle + | + \rangle - \frac{e^{i\pi/3}}{\sqrt{20}} \langle + | - \rangle + \frac{3}{\sqrt{20}} \langle - | + \rangle - \frac{e^{i\pi/3}}{\sqrt{20}} \langle - | - \rangle \right|^2$$

$$= \left(\frac{3 - e^{i\pi/3}}{\sqrt{20}} \right) \left(\frac{3 - e^{-i\pi/3}}{\sqrt{20}} \right)$$

$$\frac{9}{20} - \frac{3e^{-i\pi/3}}{20} - \frac{3e^{i\pi/3}}{20} + \frac{1}{20}$$

$$\frac{10}{20} - \frac{3}{10}\left(\frac{e^{i\pi/3}+e^{-i\pi/3}}{2}\right)$$

$$\frac{10}{20} - \frac{3}{10}\cos\frac{\pi}{3}$$

$$\frac{10}{20} - \frac{3}{20} = \underline{\frac{7}{20}}$$

$${}_x\langle -| = \frac{1}{\sqrt{2}}\langle +| - \frac{1}{\sqrt{2}}\langle -|$$

$$|\psi_3\rangle = \frac{3}{\sqrt{10}}|+\rangle - \frac{e^{i\pi/3}}{\sqrt{10}}|-\rangle$$

$$P_{3,-x} = |{}_x\langle -|\psi_3\rangle|^2$$

$$= \left|\left(\frac{1}{\sqrt{2}}\langle+| - \frac{1}{\sqrt{2}}\langle-|\right)\left(\frac{3}{\sqrt{10}}|+\rangle - \frac{e^{i\pi/3}}{\sqrt{10}}|-\rangle\right)\right|^2$$

$$= \left|\frac{3}{\sqrt{20}}\langle+|+\rangle - \frac{e^{i\pi/3}}{\sqrt{20}}\langle+|-\rangle - \frac{3}{\sqrt{20}}\langle-|+\rangle + \frac{e^{i\pi/3}}{\sqrt{20}}\langle-|-\rangle\right|^2$$

$$= \left|\frac{3}{\sqrt{20}} + \frac{e^{i\pi/3}}{\sqrt{20}}\right|^2$$

$$= \left(\frac{3}{\sqrt{20}} + \frac{e^{i\pi/3}}{\sqrt{20}}\right)\left(\frac{3}{\sqrt{20}} + \frac{e^{-i\pi/3}}{\sqrt{20}}\right)$$

$$= \left(\frac{9}{(\sqrt{20})^2} + \frac{1}{(\sqrt{20})^2} + \frac{3e^{i\pi/3}+3e^{-i\pi/3}}{20}\right)$$

$$= \frac{10}{20} + \frac{3}{10}\left(\frac{e^{i\pi/3}+e^{-i\pi/3}}{2}\right)$$

$$= \frac{10}{20} + \frac{3}{10}\cos\frac{\pi}{3} = \underline{\frac{13}{20}}$$

$${}_y\langle +| = \frac{1}{\sqrt{2}}\langle +| - \frac{i}{\sqrt{2}}\langle -|$$

$$P_{3,+y} = |{}_y\langle +|\psi_3\rangle|^2$$

$$= \left|\left(\frac{1}{\sqrt{2}}\langle +| - \frac{i}{\sqrt{2}}\langle -|\right)\left(\frac{3}{\sqrt{10}}|+\rangle - \frac{e^{i\pi/3}}{\sqrt{10}}|-\rangle\right)\right|^2$$

$$= \left|\frac{3}{\sqrt{20}}\langle +|+\rangle - \frac{e^{i\pi/3}}{\sqrt{20}}\langle +|-\rangle - \frac{3i}{\sqrt{20}}\langle -|+\rangle + \frac{ie^{i\pi/3}}{\sqrt{20}}\langle -|-\rangle\right|^2$$

$$= \left|\frac{3 + ie^{i\pi/3}}{\sqrt{20}}\right|^2$$

$$= \left(\frac{3 + ie^{i\pi/3}}{\sqrt{20}}\right)\left(\frac{3 - ie^{-\pi/3 i}}{\sqrt{20}}\right)$$

$$= \frac{1}{20}\left(9 - 3ie^{-\pi/3 i} + 3ie^{\pi/3 i} + 1\right)$$

$$= \frac{10}{20} - \frac{3}{10}\left(\frac{e^{i\pi/3} - e^{-i\pi/3}}{2i}\right)$$

$$= \frac{10}{20} - \frac{3}{10}\sin\frac{\pi}{3}$$

$$= \frac{10}{20} - \frac{3}{10}\frac{\sqrt{3}}{2}$$

$$= \frac{10 - 3\sqrt{3}}{20} = 0.24$$

$${}_y\langle -| = \frac{1}{\sqrt{2}}\left(\langle +| + i\langle -|\right)$$

$$P_{3,-y} = |{}_y\langle -|\psi_3\rangle|^2$$

$$= \left|\left(\frac{1}{\sqrt{2}}\langle +| + \frac{i}{\sqrt{2}}\langle -|\right)\left(\frac{3}{\sqrt{10}}|+\rangle - \frac{e^{i\pi/3}}{\sqrt{10}}|-\rangle\right)\right|^2$$

$$= \left|\frac{3}{\sqrt{20}}\langle +|+\rangle - \frac{e^{i\pi/3}}{\sqrt{20}}\langle +|-\rangle + \frac{3i}{\sqrt{20}}\langle -|+\rangle - \frac{ie^{i\pi/3}}{\sqrt{20}}\langle -|-\rangle\right|^2$$

$$= \left| \frac{3 - ie^{i\pi/3}}{\sqrt{20}} \right|^2$$

$$= \left| \left(\frac{3}{\sqrt{20}} - \frac{ie^{i\pi/3}}{\sqrt{20}} \right) \right| \left(\frac{3}{\sqrt{20}} + \frac{ie^{-i\pi/3}}{\sqrt{20}} \right)$$

$$= \frac{9}{20} + \frac{1}{20} - \frac{3ie^{i\pi/3}}{20} + \frac{3i}{20} e^{-i\pi/3}$$

$$= \frac{10}{20} - \frac{3}{10} \frac{ie^{i\pi/3}}{2} + \frac{3}{10} \frac{ie^{-i\pi/3}}{2}$$

$$= \frac{10}{20} + \frac{3}{10} \frac{e^{i\pi/3}}{2i} - \frac{3}{10} \frac{e^{-i\pi/3}}{2i}$$

$$= \frac{10}{20} + \frac{3}{10} \left(\frac{e^{i\pi/3} - e^{-i\pi/3}}{2i} \right)$$

$$= \frac{10}{20} + \frac{3}{10} \sin \frac{\pi}{3}$$

$$= \frac{10}{20} + \frac{3}{10} \frac{\sqrt{3}}{2}$$

$$= \frac{10 + 3\sqrt{3}}{20}$$

$$= 0.76$$

c) Write each normalized state in matrix (vector) notation. For that you first need to represent the basis states

Ans $\quad |+\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$|-\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\psi_1\rangle = \frac{3}{5}|+\rangle + \frac{4}{5}|-\rangle$$

$$= \frac{3}{5}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{4}{5}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3/5 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 4/5 \end{pmatrix}$$

$$= \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$$

$$|\psi_1\rangle = \frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{5}}|+\rangle + \frac{2i}{\sqrt{5}}|-\rangle$$

$$= \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{2i}{\sqrt{5}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{5} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2i/\sqrt{5} \end{pmatrix}$$

$$= \begin{pmatrix} 1/\sqrt{5} \\ 2i/\sqrt{5} \end{pmatrix}$$

$$|\psi_2\rangle = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ 2i \end{pmatrix}$$

$$\psi_3 = \frac{3}{\sqrt{10}}|+\rangle - \frac{e^{i\pi/3}}{\sqrt{10}}|-\rangle$$

$$= \frac{3}{\sqrt{10}}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \frac{e^{i\pi/3}}{\sqrt{10}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{10}}\begin{pmatrix} 3 \\ -e^{i\pi/3} \end{pmatrix}$$

3. Consider the three quantum states

$$|\psi_1\rangle = \frac{1}{\sqrt{3}}|+\rangle + \frac{i\sqrt{2}}{\sqrt{3}}|-\rangle$$

$$|\psi_2\rangle = \frac{1}{\sqrt{5}}|+\rangle - \frac{2}{\sqrt{5}}|-\rangle$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}|+\rangle + e^{i\pi/4}|-\rangle$$

Use bra-ket notation (not matrix notation) to solve the following problems. Note that $\langle +|+\rangle = 1$, $\langle -|-\rangle = 1$ and $\langle +|-\rangle = 0$

a. For each of the $|\psi_i\rangle$ above, find the normalized vector $|\phi_i\rangle$ that is orthogonal to it.

Ans
$$|\psi_1\rangle = \frac{1}{\sqrt{3}}|+\rangle + \frac{i\sqrt{2}}{\sqrt{3}}|-\rangle$$

$$|\phi_1\rangle = c_1|+\rangle + c_2|-\rangle$$

$$\langle \phi_1| = c_1^*\langle +| + c_2^*\langle -|$$

$$\langle \phi_1|\psi_1\rangle = (c_1^*\langle +| + c_2^*\langle -|)\left(\frac{1}{\sqrt{3}}|+\rangle + \frac{i\sqrt{2}}{\sqrt{3}}|-\rangle\right)$$

$$0 = \frac{c_1^*}{\sqrt{3}}\langle +|+\rangle + \frac{c_1 i\sqrt{2}}{\sqrt{3}}\langle +|-\rangle + \frac{c_2^*}{\sqrt{3}}\langle -|+\rangle + \frac{c_2^* i\sqrt{2}}{\sqrt{3}}\langle -|-\rangle$$

$$0 = \frac{c_1^*}{\sqrt{3}} + \frac{c_2^* i\sqrt{2}}{\sqrt{3}}$$

$$c_1^* = -i\sqrt{2}\,c_2^*$$

$$|c_1|^2 + |c_2|^2 = 1$$

$$|C_2|^2 = \frac{C_1^* i}{\sqrt{2}} \times -\frac{C_1 i}{\sqrt{2}}$$

$$= \frac{C_1^2}{2}$$

$$|C_1|^2 + |C_2|^2 = 1$$

$$|C_1|^2 + \frac{|C_1|^2}{2} = 1$$

$$\frac{3|C_1|^2}{2} = 1$$

$$|C_1| = \sqrt{\frac{2}{3}}$$

$$C_2^* = \frac{C_1^*}{\sqrt{2}} i$$

$$= \sqrt{\frac{2}{3}} \times \frac{1}{\sqrt{2}} i$$

$$C_2^* = \frac{i}{\sqrt{3}}$$

$$C_2 = -\frac{i}{\sqrt{3}}$$

$$|\varphi_1\rangle = \sqrt{\frac{2}{3}}\, |+\rangle - \frac{i}{\sqrt{3}}\, |-\rangle$$

$$|\psi_2\rangle = \frac{1}{\sqrt{5}} |+\rangle - \frac{2}{\sqrt{5}} |-\rangle$$

$$|\varphi_2\rangle = C_1 |+\rangle + C_2 |-\rangle$$

$$\langle\varphi_2| = C_1^* \langle +| + C_2^* \langle -|$$

$$\langle \varphi_2 | \psi_2 \rangle = 0$$

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$$(C_1^* \langle +| + C_2^* \langle -|)\left(\frac{1}{\sqrt{5}}|+\rangle - \frac{2}{\sqrt{5}}|-\rangle\right) = 0$$

$$\frac{C_1^*}{\sqrt{5}}\langle +|+\rangle - \frac{2}{\sqrt{5}} C_1^* \langle +|-\rangle + \frac{1}{\sqrt{5}} C_2^* \langle -|+\rangle - \frac{2C_2^*}{\sqrt{5}}\langle -|-\rangle = 0$$

$$\frac{C_1^*}{\sqrt{5}} - \frac{2C_2^*}{\sqrt{5}} = 0$$

$$C_1^* = 2C_2^*$$

$$|C_1|^2 + |C_2|^2 = 1$$

$$C_2^2 = \frac{1}{4}C_1^2$$

$$|C_1|^2 + |C_2|^2 = 1$$

$$|C_1|^2 + \frac{C_1^2}{4} = 1$$

$$\frac{5}{4}C_1^2 = 1$$

$$C_1 = \sqrt{\frac{4}{5}}$$

$$C_1 = \frac{2}{\sqrt{5}}$$

$$C_1^* = 2C_2^*$$

$$\frac{2}{\sqrt{5}} = 2C_2^*$$

$$C_2^* = \frac{1}{\sqrt{5}}$$

$$C_2 = \frac{1}{\sqrt{5}}$$

$$\Phi_2 = \frac{2}{\sqrt{5}}|+\rangle + \frac{1}{\sqrt{5}}|-\rangle$$

$$|\psi_3\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{e^{i\pi/4}}{\sqrt{2}}|-\rangle$$

$$\langle\phi_3| = c_1^* \langle +| + c_2^* \langle -|$$

$$\langle\phi_3|\psi_3\rangle = (c_1^* \langle +| + c_2^* \langle -|)\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{e^{i\pi/4}}{\sqrt{2}}|-\rangle\right)$$

$$0 = \frac{c_1^*}{\sqrt{2}} + \frac{c_2^* e^{i\pi/4}}{\sqrt{2}}$$

$$c_1^* = -c_2^* e^{i\pi/4}$$

$$c_2^* = -c_1^* e^{-i\pi/4}$$

$$c_2^2 = -c_1^* e^{-i\pi/4} \times -c_1 e^{i\pi/4}$$

$$c_2^2 = c_1^2$$

$$c_1^2 + c_2^2 = 1$$

$$c_1^2 + c_1^2 = 1$$

$$2c_1^2 = 1$$

$$c_1 = \frac{1}{\sqrt{2}}$$

$$c_2^* = -\frac{1}{\sqrt{2}} e^{-i\pi/4}$$

$$c_2 = -\frac{1}{\sqrt{2}} e^{i\pi/4}$$

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}} e^{i\pi/4}|-\rangle$$

b) Calculate the inner product $\langle \psi_i | \psi_j \rangle$ for i and $j = 1, 2, 3$

Ans $\langle \psi_1 | \psi_1 \rangle = \left(\frac{1}{\sqrt{3}} \langle +| - \frac{i\sqrt{2}}{\sqrt{3}} \langle -| \right) \left(\frac{1}{\sqrt{3}} |+\rangle + \frac{i\sqrt{2}}{\sqrt{3}} |-\rangle \right)$

$= \frac{1}{3} + \frac{2}{3} = 1$

$\langle \psi_1 | \psi_2 \rangle = \left(\frac{1}{\sqrt{3}} \langle +| - \frac{i\sqrt{2}}{\sqrt{3}} \langle -| \right) \left(\frac{1}{\sqrt{5}} |+\rangle - \frac{2}{\sqrt{5}} |-\rangle \right)$

$= \frac{1}{\sqrt{15}} + \frac{2\sqrt{2}\, i}{\sqrt{15}}$

$\langle \psi_1 | \psi_3 \rangle = \left(\frac{1}{\sqrt{3}} \langle +| - \frac{i\sqrt{2}}{\sqrt{3}} \langle -| \right) \left(\frac{1}{\sqrt{2}} |+\rangle + \frac{e^{i\pi/4}}{\sqrt{2}} |-\rangle \right)$

$= \frac{1}{\sqrt{6}} - \frac{i\sqrt{2}}{\sqrt{6}} e^{i\pi/4}$

$= \frac{1}{\sqrt{6}} - \frac{i\sqrt{2}}{\sqrt{6}} \left(\cos\frac{\pi}{4} + i \sin\frac{\pi}{4} \right)$

$= \frac{1}{\sqrt{6}} - \frac{i\sqrt{2}}{\sqrt{6}} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right)$

$= \frac{1}{\sqrt{6}} - \frac{i}{\sqrt{6}} + \frac{1}{\sqrt{6}}$

$= \frac{2 - i}{\sqrt{6}}$

$\langle \psi_2 | \psi_1 \rangle = \left(\frac{1}{\sqrt{5}} \langle +| - \frac{2}{\sqrt{5}} \langle -| \right) \left(\frac{1}{\sqrt{3}} |+\rangle + \frac{i\sqrt{2}}{\sqrt{3}} |-\rangle \right)$

$$= \frac{1}{\sqrt{15}} - \frac{2\sqrt{2}\, i}{\sqrt{15}}$$

$$\langle \psi_2 | \psi_2 \rangle = \left(\frac{1}{\sqrt{5}} \langle + | - \frac{2}{\sqrt{5}} \langle - | \right) \left(\frac{1}{\sqrt{5}} |+\rangle - \frac{2}{\sqrt{5}} |-\rangle \right)$$

$$= \frac{1}{5} + \frac{4}{5} = 1$$

$$\langle \psi_2 | \psi_3 \rangle = \left(\frac{1}{\sqrt{5}} \langle + | - \frac{2}{\sqrt{5}} \langle - | \right) \left(\frac{1}{\sqrt{2}} |+\rangle + \frac{e^{i\pi/4}}{\sqrt{2}} |-\rangle \right)$$

$$= \frac{1}{\sqrt{10}} - \frac{\sqrt{2}}{\sqrt{5}} e^{i\pi/4}$$

$$= \frac{1}{\sqrt{10}} - \frac{\sqrt{2}}{\sqrt{5}} \left(\cos\frac{\pi}{4} + i \sin\pi/4 \right)$$

$$= \frac{1}{\sqrt{10}} - \frac{\sqrt{2}}{\sqrt{5}} \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right)$$

$$= \frac{1}{\sqrt{10}} - \frac{\sqrt{2}}{\sqrt{10}} - \frac{\sqrt{2}}{\sqrt{10}} i$$

$$= \frac{1}{\sqrt{10}} \left(1 - \sqrt{2} - i\sqrt{2} \right)$$

$$\langle \psi_3 | \psi_1 \rangle = \left(\frac{1}{\sqrt{2}} \langle + | + \frac{e^{-i\pi/4}}{\sqrt{2}} \langle - | \right) \left(\frac{1}{\sqrt{3}} |+\rangle + \frac{i\sqrt{2}}{\sqrt{3}} |-\rangle \right)$$

$$= \frac{1}{\sqrt{6}} + \frac{i\sqrt{2}}{\sqrt{6}} e^{-i\pi/4}$$

$$= \frac{1}{\sqrt{6}} + \frac{i\sqrt{2}}{\sqrt{6}} \left[\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\pi/4\right) \right]$$

$$= \frac{1}{\sqrt{6}} + \frac{i\sqrt{2}}{\sqrt{6}} \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right)$$

$$= \frac{1}{\sqrt{6}} + \frac{i}{\sqrt{6}} + \frac{1}{\sqrt{6}}$$

$$= \frac{2}{\sqrt{6}} + \frac{i}{\sqrt{6}}$$

$$\langle \psi_3 | \psi_2 \rangle = \left(\frac{1}{\sqrt{2}} \langle + | + \frac{e^{-i\pi/4}}{\sqrt{2}} \langle - | \right) \left(\frac{1}{\sqrt{5}} |+\rangle - \frac{2}{\sqrt{5}} |-\rangle \right)$$

$$= \frac{1}{\sqrt{10}} - \frac{2}{\sqrt{10}} e^{-i\pi/4}$$

$$= \frac{1}{\sqrt{10}} - \frac{2}{\sqrt{10}} \left[\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\pi/4\right) \right]$$

$$= \frac{1}{\sqrt{10}} - \frac{2}{\sqrt{10}} \left[\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}} \right]$$

$$= \frac{1}{\sqrt{10}} - \frac{\sqrt{2}}{\sqrt{10}} + \frac{\sqrt{2}\, i}{\sqrt{10}}$$

$$= \frac{1}{\sqrt{10}} \left(1 - \sqrt{2} + \sqrt{2}\, i \right)$$

$$\langle \psi_3 | \psi_3 \rangle = \left(\frac{1}{\sqrt{2}} \langle + | + \frac{e^{-i\pi/4}}{\sqrt{2}} \langle - | \right) \left(\frac{1}{\sqrt{2}} |+\rangle + \frac{e^{i\pi/4}}{\sqrt{2}} |-\rangle \right)$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

4. Consider a quantum system with an observable A that has three possible measurement results:

a_1, a_2 and a_3

Normalize where necessary

a) Write down the three kets $|a_1\rangle$, $|a_2\rangle$ and $|a_3\rangle$ corresponding to these possible results using matrix notation

Ans $|a_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$|a_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$|a_3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

b) The system is prepared in the state

$|\psi\rangle = 1|a_1\rangle - 2|a_2\rangle + 5|a_3\rangle$

Write this state in matrix notation and calculate the probabilities of all possible measurement results of the observable A. Plot a histogram of the predicted measurement results

Ans $|\psi\rangle = |a_1\rangle - 2|a_2\rangle + 5|a_3\rangle$

$$= \begin{pmatrix}1\\0\\0\end{pmatrix} - 2\begin{pmatrix}0\\1\\0\end{pmatrix} + 5\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\0\end{pmatrix} - \begin{pmatrix}0\\2\\0\end{pmatrix} + \begin{pmatrix}0\\0\\5\end{pmatrix}$$

$$|\psi\rangle = \begin{pmatrix}1\\-2\\5\end{pmatrix}$$

$$|\psi\rangle = c\begin{pmatrix}1\\-2\\5\end{pmatrix}$$

$$\langle\psi| = c^*\begin{pmatrix}1 & -2 & 5\end{pmatrix}$$

$$\langle\psi|\psi\rangle = c^*\begin{pmatrix}1 & -2 & 5\end{pmatrix} c\begin{pmatrix}1\\-2\\5\end{pmatrix}$$

$$1 = c^2(1+4+25)$$

$$1 = c^2(30)$$

$$c = \frac{1}{\sqrt{30}}$$

$$\langle a_1|\psi\rangle = \begin{pmatrix}1 & 0 & 0\end{pmatrix}\frac{1}{\sqrt{30}}\begin{pmatrix}1\\-2\\5\end{pmatrix}$$

$$= \frac{1}{\sqrt{30}}$$

$$P(a_1||\psi\rangle) = (\langle a_1|\psi\rangle)^2$$

$$= \frac{1}{30}$$

$$\langle a_2 | \psi \rangle = (0\ 1\ 0)\ \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$$

$$= -\frac{2}{\sqrt{30}}$$

$$P(a_2 | |\psi\rangle) = |\langle a_2 | \psi \rangle|^2$$

$$= \left| -\frac{2}{\sqrt{30}} \right|^2$$

$$= \frac{4}{(\sqrt{30})^2} = \frac{4}{30}$$

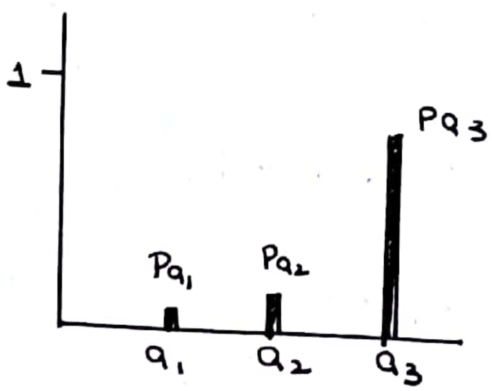
$$\langle a_3 | \psi \rangle = (0\ 0\ 1)\ \frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ -2 \\ +5 \end{pmatrix}$$

$$= \frac{5}{\sqrt{30}}$$

$$P(a_3 | |\psi\rangle) = |\langle a_3 | \psi \rangle|^2$$

$$= \left| \frac{5}{\sqrt{30}} \right|^2$$

$$= \frac{25}{30}$$



c) In a different experiment, the system is prepared in the state

$$|\Psi\rangle = 2|a_1\rangle + 3i\,|a_2\rangle$$

Write the state in matrix notation and calculate the probabilities of all possible measurement results of the observable A. Plot a histogram of the predicted measurement results

Ans

$$|\Psi\rangle = 2\begin{pmatrix}1\\0\\0\end{pmatrix} + 3i\begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$= \begin{pmatrix}2\\0\\0\end{pmatrix} + \begin{pmatrix}0\\3i\\0\end{pmatrix}$$

$$|\Psi\rangle = \begin{pmatrix}2\\3i\\0\end{pmatrix}$$

$$|\Psi\rangle = c\begin{pmatrix}2\\3i\\0\end{pmatrix}$$

$$\langle\Psi| = c^*\begin{pmatrix}2 & -3i & 0\end{pmatrix}$$

$$\langle\Psi|\Psi\rangle = c^*\begin{pmatrix}2 & 3i & 0\end{pmatrix} c\begin{pmatrix}2\\3i\\0\end{pmatrix}$$

$$1 = c^2(4+9)$$

$$c = \frac{1}{\sqrt{13}}$$

$$\langle a_1|\psi\rangle = (1\ 0\ 0)\frac{1}{\sqrt{13}}\begin{pmatrix}2\\3i\\0\end{pmatrix}$$

$$= \frac{2}{\sqrt{13}}$$

$$P(a_1||\psi\rangle) = \frac{4}{13}$$

$$\langle a_2|\psi\rangle = (0\ 1\ 0)\frac{1}{\sqrt{13}}\begin{pmatrix}2\\3i\\0\end{pmatrix}$$

$$= \frac{3i}{\sqrt{13}}$$

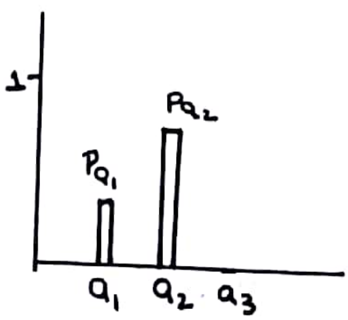
$$P(a_2||\psi\rangle) = \frac{9}{13}$$

$$\langle a_3|\psi\rangle = (0\ 0\ 1)\frac{1}{\sqrt{13}}\begin{pmatrix}2\\3i\\0\end{pmatrix}$$

$$= 0$$

$$P(a_3||\psi\rangle) = |\langle a_3|\psi\rangle|^2$$

$$= 0$$



5. Consider a quantum system described by the basis $|a_1\rangle$, $|a_2\rangle$ and $|a_3\rangle$. The system is initially in a state

$$|\psi_i\rangle = \frac{1}{\sqrt{3}}|a_1\rangle + \sqrt{\frac{2}{3}}|a_2\rangle$$

Find the probability that the system is measured to be in the final state

$$|\psi_f\rangle = \frac{1+i}{\sqrt{3}}|a_1\rangle + \sqrt{\frac{1}{6}}|a_2\rangle + \sqrt{\frac{1}{6}}|a_3\rangle$$

Ans $\langle\psi_f|\psi_i\rangle = \left(\frac{1-i}{\sqrt{3}}\langle a_1| + \sqrt{\frac{1}{6}}\langle a_2| + \sqrt{\frac{1}{6}}\langle a_3|\right)\left(\frac{1}{\sqrt{3}}|a_1\rangle + \sqrt{\frac{2}{3}}|a_2\rangle\right)$

$$= \frac{1-i}{3} + \sqrt{\frac{2}{18}}$$

$$= \frac{1}{3} - \frac{i}{3} + \sqrt{\frac{2}{18}}$$

$$= \frac{1}{3} - \frac{i}{3} + \sqrt{\frac{1}{9}}$$

$$= \frac{1}{3} + \frac{1}{3} - \frac{i}{3}$$

$$= \frac{2}{3} - \frac{i}{3}$$

$$P(\psi_f||\psi_i\rangle) = |\langle\psi_f|\psi_i\rangle|^2$$

$$= \left(\frac{2}{3} - \frac{i}{3}\right)\left(\frac{2}{3} + \frac{i}{3}\right)$$

$$= \frac{4}{9} + \frac{1}{9} = \frac{5}{9}$$

6. The spin components of a beam of atoms prepared in the state $|\psi_{in}\rangle$ are measured and the following experimental probabilities are obtained

$P_+ = \frac{1}{2}$ $\quad P_{+x} = \frac{3}{4}$ $\quad P_{+y} = 0.067$

$P_- = \frac{1}{2}$ $\quad P_{-x} = \frac{1}{4}$ $\quad P_{-y} = 0.933$

From the experimental data estimate the input state. This mean write your best guess of $|\psi_{in}\rangle$ represented in a suitable basis.

Ans $\quad P_+ = \frac{1}{2}$

$$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle$$

$$P_+ = |\langle + |\psi\rangle|^2$$

$$= |\langle + |(\alpha|+\rangle + \beta|-\rangle)|^2$$

$$= |\alpha\langle +|+\rangle + \beta\langle +|-\rangle|^2$$

$$\frac{1}{2} = |\alpha|^2$$

$$\alpha = \frac{1}{\sqrt{2}}$$

$$P_- = |\langle - |\psi\rangle|^2$$

$$= |\langle - |(\alpha|+\rangle + \beta|-\rangle)|^2$$

$$\frac{1}{2} = |\beta|^2$$

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$$\beta = \frac{1}{\sqrt{2}} e^{i\varphi}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} e^{i\varphi} |-\rangle$$

$$P_{+x} = \frac{3}{4}$$

$$|+\rangle_x = \frac{1}{\sqrt{2}} [\,|+\rangle + |-\rangle\,]$$

$$_x\langle +| = \frac{1}{\sqrt{2}} \langle +| + \frac{1}{\sqrt{2}} \langle -|$$

$$_x\langle +|\psi\rangle = \left(\frac{1}{\sqrt{2}} \langle +| + \frac{1}{\sqrt{2}} \langle -| \right) \left(\frac{1}{\sqrt{2}} |+\rangle + \frac{1}{\sqrt{2}} e^{i\varphi} |-\rangle \right.$$

$$= \frac{1}{2} + \frac{1}{2} e^{i\varphi}$$

$$P_{+x} = |{}_x\langle + |\psi\rangle|^2$$

$$= \left(\frac{1}{2} + \frac{1}{2} e^{i\varphi} \right) \left(\frac{1}{2} + \frac{1}{2} e^{-i\varphi} \right)$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} e^{i\varphi} + \frac{1}{4} e^{-i\varphi}$$

$$= \frac{1}{2} + \frac{1}{2} \left(\frac{e^{i\varphi} + e^{-i\varphi}}{2} \right)$$

$$\frac{3}{4} = \frac{1}{2} + \frac{1}{2} \cos\varphi$$

$$\frac{3}{4} - \frac{1}{2} = \frac{1}{2} \cos\varphi$$

$$+\frac{1}{4} = \frac{1}{2}\cos\varphi$$

$$\cos\varphi = +\frac{1}{2}$$

$$\varphi = \frac{\pi}{3} \quad \text{or} \quad \frac{5\pi}{3}$$

$$|+\rangle_y = \frac{1}{\sqrt{2}}|+\rangle + \frac{i}{\sqrt{2}}|-\rangle$$

$${}_y\langle +| = \frac{1}{\sqrt{2}}\langle +| - \frac{i}{\sqrt{2}}\langle -|$$

$${}_y\langle +|\psi\rangle = \left(\frac{1}{\sqrt{2}}\langle +| - \frac{i}{\sqrt{2}}\langle -|\right)\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{e^{i\varphi}}{\sqrt{2}}|-\rangle\right)$$

$$= \frac{1}{2} - \frac{i\,e^{i\varphi}}{2}$$

$$P_{+y} = |\langle +|\psi\rangle|^2$$

$$0.067 = \left(\frac{1}{2} - \frac{ie^{i\varphi}}{2}\right)\left(\frac{1}{2} + \frac{ie^{-i\varphi}}{2}\right)$$

$$= \frac{1}{4} + \frac{1}{4} - \frac{i}{4}e^{i\varphi} + \frac{ie^{-i\varphi}}{4}$$

$$0.067 = \frac{2}{4} + \frac{e^{i\varphi}}{4i} - \frac{e^{-i\varphi}}{4i}$$

$$0.067 = \frac{1}{2} + \frac{1}{2}\left(\frac{e^{i\varphi} - e^{-i\varphi}}{2i}\right)$$

$$0.067 = \frac{1}{2} + \frac{1}{2}\sin\varphi$$

$$\varphi = -\frac{\pi}{3} = -\frac{\pi}{3} + 2\pi = \frac{5\pi}{3}$$

$$\varphi = \frac{5\pi}{3}$$

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$$|\Psi\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}} e^{i\frac{5\pi}{3}}|-\rangle$$