Assignment #2

(Solution)

(1) (a) $\langle \phi | \doteq \begin{bmatrix} 2 & i & 2+3i \end{bmatrix}$

(b) $\langle \phi | \psi \rangle \doteq \begin{bmatrix} 2 & i & 2+3i \end{bmatrix} \begin{bmatrix} -3i \\ 2+i \\ 4 \end{bmatrix} = 7+8i$

(c) $|\psi\rangle|\phi\rangle \doteq \begin{bmatrix} -3i \\ 2+i \\ 4 \end{bmatrix} \begin{bmatrix} 2 \\ -i \\ 2-3i \end{bmatrix}$, and $\langle \psi | \langle \phi | \doteq$

$\begin{bmatrix} 3i & 2-i & 4 \end{bmatrix}$

$\begin{bmatrix} 2 & i & 2+3i \end{bmatrix}$.

These matrix products are not defined.

(2) (a) $|\psi\rangle \doteq \begin{bmatrix} 3i \\ -7i \end{bmatrix}$, and $|\chi\rangle \doteq \begin{bmatrix} -1 \\ 2i \end{bmatrix}$.

(b) $\langle \psi | \chi \rangle \doteq \begin{bmatrix} -3i & 7i \end{bmatrix} \begin{bmatrix} -1 \\ 2i \end{bmatrix} = -14 + 3i.$

$\langle \chi | \psi \rangle \doteq \begin{bmatrix} -1 & -2i \end{bmatrix} \begin{bmatrix} 3i \\ -7i \end{bmatrix} = -14 - 3i.$

(c) Clearly, $\langle \psi | \chi \rangle = \overline{\langle \chi | \psi \rangle}$, or $(\langle \psi | \chi \rangle)^* = \langle \chi | \psi \rangle$.

(3) (a) $|\psi\rangle\langle\chi| \doteq \begin{bmatrix} i \\ 3i \\ -1 \end{bmatrix} \begin{bmatrix} 1 & i & -5i \end{bmatrix} = \begin{bmatrix} i & -1 & 5 \\ 3i & -3 & 15 \\ -1 & -i & 5i \end{bmatrix}$.

$|\chi\rangle\langle\psi| \doteq \begin{bmatrix} 1 \\ -i \\ 5i \end{bmatrix} \begin{bmatrix} -i & -3i & -1 \end{bmatrix} = \begin{bmatrix} -i & -3i & -1 \\ -1 & -3 & i \\ 5 & 15 & -5i \end{bmatrix}$.

Clearly, $|\psi\rangle\langle\chi| \neq |\chi\rangle\langle\psi|$.

(b) $\left(\left(\begin{bmatrix} i & -1 & 5 \\ 3i & -3 & 15 \\ -1 & -i & 5i \end{bmatrix} \right)^T \right)^* = \begin{bmatrix} -i & -3i & -1 \\ -1 & -3 & i \\ 5 & 15 & -5i \end{bmatrix}$.

The answer is $|\chi\rangle\langle\psi|$ written in matrix notation.

(4)(a) $|\psi\rangle^* \doteq \begin{bmatrix} -5i \\ 2 \\ i \end{bmatrix}$, and $\langle\psi| \doteq \begin{bmatrix} -5i & 2 & i \end{bmatrix}$.

(b) $\langle\psi|\psi\rangle \doteq \begin{bmatrix} -5i & 2 & i \end{bmatrix} \begin{bmatrix} 5i \\ 2 \\ -i \end{bmatrix}$

$= 25 + 4 + 1 = 30 \neq 1$.

$\therefore |\psi\rangle$ is not normalized.

Let our normalized state be $|\psi\rangle \doteq c \begin{bmatrix} 5i \\ 2 \\ -i \end{bmatrix}$, where $c \in \mathbb{C}$.

$\langle\psi|\psi\rangle = 1 \Rightarrow c^* c \begin{bmatrix} -5i & 2 & i \end{bmatrix} \begin{bmatrix} 5i \\ 2 \\ -i \end{bmatrix} = 1 \Rightarrow \|c\|^2 = 30$

$\Rightarrow \|c\| = \frac{1}{\sqrt{30}}$. By standard convention, $c = \frac{1}{\sqrt{30}}$. Hence,

$|\psi\rangle \doteq \frac{1}{\sqrt{30}} \begin{bmatrix} 5i \\ 2 \\ -i \end{bmatrix}$. ~ Normalized state

(c) $\langle\psi|\phi\rangle \doteq \begin{bmatrix} -5i & 2 & i \end{bmatrix} \begin{bmatrix} 3 \\ 8i \\ -9i \end{bmatrix} = 9 + i \neq 0$. Hence, $|\psi\rangle$ and $|\phi\rangle$ are not orthogonal.

(5)

$$|\psi\rangle = \sum_k c_k |k\rangle$$

$$\langle i|\psi\rangle = \sum_k c_k \langle i|k\rangle$$

$$\langle i|\psi\rangle = \sum_k c_k \delta_{ik}$$

$$\Rightarrow c_i = \langle i|\psi\rangle \qquad \text{Q.E.D.}$$