

Assignment #3

(Solution)

(1)(a) Let $|\lambda\rangle$ be an eigenvector of \hat{U} , and let λ be the corresponding eigenvalue.

$$\hat{U}|\lambda\rangle = \lambda|\lambda\rangle$$

$$\Rightarrow \langle\lambda|\hat{U}^\dagger = \langle\lambda|\lambda^*$$

$$\langle\lambda|\hat{U}^\dagger\hat{U}|\lambda\rangle = \langle\lambda|\lambda^*\lambda|\lambda\rangle$$

$$\langle\lambda|\hat{I}|\lambda\rangle = \|\lambda\|^2 \langle\lambda|\lambda\rangle$$

$$\langle\lambda|\lambda\rangle = \|\lambda\|^2 \langle\lambda|\lambda\rangle$$

$$\|\lambda\|^2 = 1$$

$$\|\lambda\| = 1$$

$$\Rightarrow \lambda = e^{i\phi}, \quad 0 \leq \phi < 2\pi. \quad \text{Q.E.D.}$$

(b)

$$(\hat{U}_1\hat{U}_2)^\dagger(\hat{U}_1\hat{U}_2) = \hat{U}_2^\dagger\hat{U}_1^\dagger\hat{U}_1\hat{U}_2 = \hat{U}_2^\dagger\hat{I}\hat{U}_2 = \hat{U}_2^\dagger\hat{U}_2 = \hat{I}.$$

$$(\hat{U}_1\hat{U}_2)(\hat{U}_1\hat{U}_2)^\dagger = \hat{U}_1\hat{U}_2\hat{U}_2^\dagger\hat{U}_1^\dagger = \hat{U}_1\hat{I}\hat{U}_1^\dagger = \hat{U}_1\hat{U}_1^\dagger = \hat{I}.$$

$\Rightarrow \hat{U}_1\hat{U}_2$ is unitary.

(c)

$$|\phi\rangle = \hat{U}|\psi\rangle$$

$$\begin{aligned} \langle\phi|\phi\rangle &= (\hat{U}|\psi\rangle)^\dagger(\hat{U}|\psi\rangle) = \langle\psi|\hat{U}^\dagger\hat{U}|\psi\rangle = \langle\psi|\hat{I}|\psi\rangle \\ &= \langle\psi|\psi\rangle = 1. \end{aligned}$$

Hence, $|\phi\rangle = \hat{U}|\psi\rangle$ and $|\psi\rangle$ have the same norm.

(2)(a) Let λ be an eigenvalue of \hat{A} , and let the corresponding eigenvector be $|\lambda\rangle$.

$$\hat{A}|\lambda\rangle = \lambda|\lambda\rangle$$

$$\boxed{\langle\lambda|\hat{A}|\lambda\rangle = \lambda\langle\lambda|\lambda\rangle} \rightarrow \textcircled{A}$$

$$\langle \lambda | \hat{A}^\dagger = \langle \lambda | \lambda^*$$

$$\langle \lambda | \hat{A}^\dagger | \lambda \rangle = \lambda^* \langle \lambda | \lambda \rangle$$

$$\textcircled{B} \leftarrow \boxed{\langle \lambda | \hat{A} | \lambda \rangle = \lambda^* \langle \lambda | \lambda \rangle} \quad (\because \hat{A} \text{ is Hermitian})$$

Using \textcircled{A} and \textcircled{B} , we can write

$$\lambda^* \langle \lambda | \lambda \rangle = \lambda \langle \lambda | \lambda \rangle$$

$$\lambda^* = \lambda.$$

Hence, every eigenvalue of a Hermitian operator is real.

$$(b) (\hat{A}_1 \hat{A}_2)^\dagger = \hat{A}_2^\dagger \hat{A}_1^\dagger = \hat{A}_2 \hat{A}_1 \neq \hat{A}_1 \hat{A}_2.$$

Hence, $\hat{A}_1 \hat{A}_2$ is not Hermitian.

(c) Let $|\lambda_1\rangle$ and $|\lambda_2\rangle$ be the eigenvectors of \hat{A} , and let λ_1 and λ_2 be the corresponding eigenvalues, where $\lambda_1 \neq \lambda_2$.

$$\hat{A} |\lambda_1\rangle = \lambda_1 |\lambda_1\rangle$$

$$\boxed{\langle \lambda_2 | \hat{A} | \lambda_1 \rangle = \lambda_1 \langle \lambda_2 | \lambda_1 \rangle}$$

$$\langle \lambda_2 | \hat{A} = \langle \lambda_2 | \lambda_2 \quad (\because \hat{A} \text{ is Hermitian, and } \lambda_2 \text{ is real.})$$

$$\boxed{\langle \lambda_2 | \hat{A} | \lambda_1 \rangle = \lambda_2 \langle \lambda_2 | \lambda_1 \rangle}$$

Subtraction gives

$$(\lambda_1 - \lambda_2) \langle \lambda_2 | \lambda_1 \rangle = 0$$

nonzero, for $\lambda_1 \neq \lambda_2$

$$\Rightarrow \langle \lambda_2 | \lambda_1 \rangle = 0. \text{ Q.E.D.}$$

$$\begin{aligned} (3)(a) (\hat{S}^\dagger \hat{S})_{ij} &= \sum_k (\hat{S}^\dagger)_{ik} (\hat{S})_{kj} = \sum_k (\hat{S}_{ki})^\dagger (\hat{S})_{kj} = \sum_k \langle \tilde{k} | i \rangle^\dagger \langle \tilde{k} | j \rangle \\ &= \sum_k \langle i | \tilde{k} \rangle \langle \tilde{k} | j \rangle = \langle i | \sum_k |\tilde{k}\rangle \langle \tilde{k}| | j \rangle = \langle i | j \rangle = \delta_{ij}. \end{aligned}$$

$$\begin{aligned} (\hat{S} \hat{S}^\dagger)_{ij} &= \sum_k (\hat{S})_{ik} (\hat{S}^\dagger)_{kj} = \sum_k (\hat{S})_{ik} (\hat{S}_{jk})^\dagger = \sum_k \langle \tilde{i} | k \rangle \langle \tilde{j} | k \rangle^\dagger \\ &= \sum_k \langle \tilde{i} | k \rangle \langle k | \tilde{j} \rangle = \langle \tilde{i} | \sum_k |\tilde{k}\rangle \langle \tilde{k}| | \tilde{j} \rangle = \langle \tilde{i} | \tilde{j} \rangle = \delta_{ij}. \end{aligned}$$

Clearly, \hat{S} is unitary.

(b) Let \hat{R} be the operator. Then,

$$\hat{R}|H\rangle = \cos(\alpha)|H\rangle + \sin(\alpha)|V\rangle$$

$$\hat{R}|V\rangle = -\sin(\alpha)|H\rangle + \cos(\alpha)|V\rangle$$

$$\hat{R} = \begin{bmatrix} \langle H|\hat{R}|H\rangle & \langle H|\hat{R}|V\rangle \\ \langle V|\hat{R}|H\rangle & \langle V|\hat{R}|V\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

(c) The new basis is $\left\{ \frac{1}{\sqrt{2}}|H\rangle + \frac{1}{\sqrt{2}}|V\rangle, \frac{1}{\sqrt{2}}|H\rangle - \frac{1}{\sqrt{2}}|V\rangle \right\}$.

$$\hat{S} = \begin{bmatrix} \langle \tilde{1}|H\rangle & \langle \tilde{1}|V\rangle \\ \langle \tilde{2}|H\rangle & \langle \tilde{2}|V\rangle \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$\hat{\tilde{R}} = \hat{S}\hat{R}\hat{S}^{-1} = \hat{S}\hat{R}\hat{S}^\dagger$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\cos(\alpha) + \sin(\alpha)}{\sqrt{2}} & \frac{-\sin(\alpha) + \cos(\alpha)}{\sqrt{2}} \\ \frac{-\cos(\alpha) + \sin(\alpha)}{\sqrt{2}} & \frac{\sin(\alpha) + \cos(\alpha)}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

$$\begin{aligned} (4)(a) \quad \hat{A} + \hat{A}^\dagger &\rightarrow (\hat{A} + \hat{A}^\dagger)^\dagger = \hat{A}^\dagger + \hat{A}^{\dagger\dagger} = \hat{A}^\dagger + \hat{A} = \hat{A} + \hat{A}^\dagger \text{ (Hermitian)} \\ (i(\hat{A} + \hat{A}^\dagger))^\dagger &\rightarrow -i(\hat{A}^\dagger + \hat{A}^{\dagger\dagger}) = -i(\hat{A}^\dagger + \hat{A}) = -i(\hat{A} + \hat{A}^\dagger) \text{ (AntiHermitian)} \\ (i(\hat{A} - \hat{A}^\dagger))^\dagger &\rightarrow -i(\hat{A}^\dagger - \hat{A}^{\dagger\dagger}) = -i(\hat{A}^\dagger - \hat{A}) = i(\hat{A} - \hat{A}^\dagger) \text{ (Hermitian)} \end{aligned}$$

(b) $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthonormal. Thus they are linearly independent. In a 2-dimensional Hilbert space, a linearly independent set of two vectors must span the entire space. Therefore, $|\phi_1\rangle$ and $|\phi_2\rangle$ form a basis.

(c) $\hat{B}^\dagger = (|\phi_1\rangle\langle\phi_2|)^\dagger = |\phi_2\rangle\langle\phi_1| \neq |\phi_1\rangle\langle\phi_2| = \hat{B}$. Hence, \hat{B} is not Hermitian.

$$\hat{B}^2 = |\phi_1\rangle\langle\phi_2| |\phi_1\rangle\langle\phi_2| = \hat{0}. \text{ Q.E.D.}$$

$$(d) \hat{B}\hat{B}^\dagger = |\phi_1\rangle\langle\phi_2| |\phi_2\rangle\langle\phi_1| = |\phi_1\rangle\langle\phi_1|.$$

$$\hat{B}^\dagger\hat{B} = |\phi_2\rangle\langle\phi_1| |\phi_1\rangle\langle\phi_2| = |\phi_2\rangle\langle\phi_2|.$$

(e) $(\hat{B}\hat{B}^\dagger + \hat{B}^\dagger\hat{B})^\dagger = (|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|)^\dagger = |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| = \hat{B}\hat{B}^\dagger + \hat{B}^\dagger\hat{B}$. Hence, $\hat{B}\hat{B}^\dagger + \hat{B}^\dagger\hat{B}$ is Hermitian.

(f) $(\hat{B}\hat{B}^\dagger - \hat{B}^\dagger\hat{B})^\dagger = (|\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2|)^\dagger = |\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2| = \hat{B}\hat{B}^\dagger - \hat{B}^\dagger\hat{B}$. Hence, $\hat{B}\hat{B}^\dagger - \hat{B}^\dagger\hat{B}$ is Hermitian.

$$(g) \hat{C}|\phi_1\rangle = (|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|)|\phi_1\rangle = |\phi_1\rangle\langle\phi_1|\phi_1\rangle + |\phi_2\rangle\langle\phi_2|\phi_1\rangle = |\phi_1\rangle$$

$$\hat{C}|\phi_2\rangle = (|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|)|\phi_2\rangle = |\phi_1\rangle\langle\phi_1|\phi_2\rangle + |\phi_2\rangle\langle\phi_2|\phi_2\rangle = |\phi_2\rangle$$

$$(a) \hat{A}^\dagger = (|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| + |\phi_3\rangle\langle\phi_3| - i|\phi_1\rangle\langle\phi_2| - |\phi_1\rangle\langle\phi_3| + i|\phi_2\rangle\langle\phi_1| - |\phi_3\rangle\langle\phi_1|)^\dagger$$

$$= |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| + |\phi_3\rangle\langle\phi_3| + i|\phi_2\rangle\langle\phi_1| - |\phi_3\rangle\langle\phi_1| - i|\phi_1\rangle\langle\phi_2| - |\phi_1\rangle\langle\phi_3|$$

$$= |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| + |\phi_3\rangle\langle\phi_3| - i|\phi_1\rangle\langle\phi_2| - |\phi_1\rangle\langle\phi_3| + i|\phi_2\rangle\langle\phi_1| - |\phi_3\rangle\langle\phi_1| = \hat{A}$$

$\therefore \hat{A}$ is Hermitian.

$$\hat{A}^2 = (|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| + |\phi_3\rangle\langle\phi_3| - i|\phi_1\rangle\langle\phi_2| - |\phi_1\rangle\langle\phi_3| + i|\phi_2\rangle\langle\phi_1| - |\phi_3\rangle\langle\phi_1|) (|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| + |\phi_3\rangle\langle\phi_3| - i|\phi_1\rangle\langle\phi_2| - |\phi_1\rangle\langle\phi_3| + i|\phi_2\rangle\langle\phi_1| - |\phi_3\rangle\langle\phi_1|)$$

$$\hat{A}^2 \neq \hat{A}.$$

$\therefore \hat{A}$ is not a projection operator.

(b)

$$\hat{A} = \begin{bmatrix} 1 & -i & -1 \\ i & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Note: One may check again.

$$\hat{A}^2 = \begin{bmatrix} 1 & -i & -1 \\ i & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -i & -1 \\ i & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\hat{A}^2 = \begin{bmatrix} 3 & -2i & -2 \\ 2i & 2 & -i \\ -2 & i & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & -i & -1 \\ i & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Hence, \hat{A} is not a projection operator.

(5) Let

$$|\tilde{I}\rangle = \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}.$$

$$|2\rangle = \begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix} - \frac{[1 \ 0 \ -i] \begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix}}{[1 \ 0 \ -i] \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}$$

$$|2\rangle = \begin{bmatrix} -1 \\ i \\ 1 \end{bmatrix} + \left(\frac{1}{2} + \frac{1}{2}i\right) \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}$$

$$|2\rangle = \begin{bmatrix} -0.5 + 0.5i \\ i \\ 0.5 + 0.5i \end{bmatrix}$$

$$|3\rangle = \begin{bmatrix} 0 \\ -1 \\ 1+i \end{bmatrix} - \frac{[1 \ 0 \ -i] \begin{bmatrix} 0 \\ -1 \\ 1+i \end{bmatrix}}{[1 \ 0 \ -i] \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}$$

$$= \begin{bmatrix} -0.5 - 0.5i - i \ 0.5 - 0.5i \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1+i \end{bmatrix} - \frac{[-0.5 - 0.5i \ -i \ 0.5 - 0.5i] \begin{bmatrix} -0.5 + 0.5i \\ i \\ 0.5 + 0.5i \end{bmatrix}}{[1 \ 0 \ -i] \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}$$

$$|3\rangle = \begin{bmatrix} 0 \\ -1 \\ 1+i \end{bmatrix} - \left(\frac{1}{2} - \frac{i}{2}\right) \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix} - \left(\frac{1}{2} + \frac{i}{2}\right) \begin{bmatrix} -0.5 + 0.5i \\ i \\ 0.5 + 0.5i \end{bmatrix}$$

$$|3\rangle = \begin{bmatrix} 0.5i \\ -0.5 - 0.5i \\ 0.5 \end{bmatrix}$$

The orthogonal vectors we wanted from the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ i \end{bmatrix}, \begin{bmatrix} -0.5 + 0.5i \\ i \\ 0.5 + 0.5i \end{bmatrix}, \begin{bmatrix} 0.5i \\ -0.5 - 0.5i \\ 0.5 \end{bmatrix} \right\}.$$

Normalizing them in the usual way, we obtain the following orthonormal set:

$$\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{i}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{2\sqrt{2}} + \frac{i}{2\sqrt{2}} \\ 0 \\ \frac{1}{2\sqrt{2}} + \frac{i}{2\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} - \frac{i}{2} \\ \frac{1}{2} \end{bmatrix} \right\}.$$